

# Packing and Covering Immersions in 4-Edge-Connected Graphs

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## Abstract

A graph  $G$  immerses  $H$  if  $H$  can be obtained from a subgraph of  $G$  by splitting off edges and removing isolated vertices. In this paper, we prove an edge-variant of the Erdős-Pósa property with respect to the immersion containment in 4-edge-connected graphs. More precisely, we prove that for every graph  $H$ , there exists a function  $f$  such that for every 4-edge-connected graph  $G$ , either  $G$  contains  $k$  pairwise edge-disjoint subgraphs each immersing  $H$ , or there exist at most  $f(k)$  edges of  $G$  intersecting all such subgraphs. The theorem is best possible in the sense that the 4-edge-connectivity cannot be replaced by the 3-edge-connectivity.

## 1 Introduction

In this paper *graphs* are finite and are permitted to have loops and parallel edges. Many questions in graph theory or combinatorial optimization can be formulated as follows. Given a set of graphs  $\mathcal{F}$  and a graph  $G$ , what is the maximum number of disjoint subgraphs in  $G$  each isomorphic to a member in  $\mathcal{F}$  or what is the minimum number of vertices that meet all such subgraphs?

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We call the former problem the packing problem and the maximum number the packing number, and we call the latter problem the covering problem and the minimum the covering number. For example, if  $\mathcal{F}$  consists of an edge, then the packing number is the maximum size of a matching and the covering number is the minimum size of a vertex cover; if  $\mathcal{F}$  is the set of cycles, the covering number is the minimum size of a feedback vertex set.

In view of combinatorial optimization, the covering problem is the dual of the packing problem. Furthermore, it is easy to see that the packing number is at most the covering number. On the other hand, it is natural to ask when the covering number can be bounded by a function of the packing number from above. In other words, we hope that the optimal solutions of the packing problem and covering problem are bounded by a function of each other.

Formally, a set of graphs  $\mathcal{F}$  has the *Erdős-Pósa property* if for every integer  $k$ , there exists a number  $f(k)$  such that for every graph  $G$ , either  $G$  contains  $k$  disjoint subgraphs each isomorphic to a member of  $\mathcal{F}$ , or there exists  $Z \subseteq V(G)$  with  $|Z| \leq f(k)$  such that  $G - Z$  does not contain a subgraph isomorphic to a member of  $\mathcal{F}$ . A classical result of Erdős and Pósa [2] states that the set of cycles has the Erdős-Pósa property. This theorem was later generalized by Robertson and Seymour in terms of graph minors.

A graph is a *minor* of another if the first can be obtained from a subgraph of the second by contracting edges. For every graph  $H$ , define  $\mathcal{M}(H)$  to be the set of graphs containing  $H$  as a minor. In particular,  $\mathcal{M}(H)$  is the set of cycles if  $H$  is the loop. Robertson and Seymour [6] proved that  $\mathcal{M}(H)$  has the Erdős-Pósa property if and only if  $H$  is planar.

A graph is a *topological minor* of another if the first can be obtained from a subgraph of the second by contracting edges incident with vertices of degree two. For every graph  $H$ , define  $\mathcal{TM}(H)$  to be the set of graphs containing  $H$  as a topological minor. Unlike graph minors, the Erdős-Pósa property for  $\mathcal{TM}(H)$  is not equivalent with the planarity of  $H$ . The author, Postle and Wollan [4] provided a characterization of graphs  $H$  in which  $\mathcal{TM}(H)$  has the Erdős-Pósa property and proved that it is NP-hard to decide whether  $\mathcal{TM}(H)$  has the Erdős-Pósa property for the input graph  $H$ .

The topological minor relation can be equivalently defined as follows. A graph  $H$  with no isolated vertices is a topological minor of another graph  $G$  if there exist an injection  $\pi_V$  from  $V(H)$  to  $V(G)$  and a function  $\pi_E$  that maps the edges  $e = uv$  of  $H$  to paths in  $G$  from  $\pi_V(u)$  to  $\pi_V(v)$  (if  $e$  is a loop with the end  $v$ , then  $\pi_E(e)$  is a cycle in  $G$  containing  $\pi_V(v)$ ) such that

$\pi_E(e_1)$  and  $\pi_E(e_2)$  are internally disjoint for distinct edges  $e_1, e_2$  of  $H$ . Note that we consider a loop as a cycle as well.

We say that a graph  $H$  (allowing isolated vertices) is an *immersion* of a graph  $G$  if the mentioned internally disjoint property is replaced by the edge-disjoint property. In other words, an  $H$ -*immersion* in  $G$  is a pair of functions  $(\pi_V, \pi_E)$  such that the following hold.

- $\pi_V$  is an injection from  $V(H)$  to  $V(G)$ .
- $\pi_E$  maps  $E(H)$  to a subgraph of  $G$  such that for every edge  $e$  of  $H$ , if  $e$  has distinct ends  $x, y$ , then  $\pi_E(e)$  is a path with ends  $\pi_V(x)$  and  $\pi_V(y)$ , and if  $e$  is the loop with end  $v$ , then  $\pi_E(e)$  is a cycle containing  $\pi_V(v)$ .
- If  $e_1, e_2$  are distinct edges of  $H$ ,  $\pi_E(e_1)$  and  $\pi_E(e_2)$  are edge-disjoint.

We say that two  $H$ -immersions  $(\pi_V, \pi_E)$  and  $(\pi'_V, \pi'_E)$  are *edge-disjoint* if the image of  $\pi_E$  is disjoint from the image of  $\pi'_E$ .

As immersions consist of edge-disjoint paths, it is reasonable to ask for packing edge-disjoint copies of immersions instead of disjoint copies. Furthermore, one vertex can meet at least two edge-disjoint copies of immersions, so it is more natural to cover these edge-disjoint subgraphs by edges instead of by vertices. This motivates an edge-variant of the Erdős-Pósa property. We say that a set  $\mathcal{F}$  of graphs has the *edge-variant of the Erdős-Pósa property* if for every integer  $k$ , there exists a number  $f(k)$  such that for every graph  $G$ , either  $G$  contains  $k$  edge-disjoint subgraphs each isomorphic to a member of  $\mathcal{F}$ , or there exists  $Z \subseteq E(G)$  with  $|Z| \leq f(k)$  such that  $G - Z$  has no subgraph isomorphic to a member of  $\mathcal{F}$ . Raymond, Sau and Thilikos [5] proved that  $\mathcal{M}(\theta_r)$  has the edge-variant of the Erdős-Pósa property, where  $\theta_r$  is the loopless graph on two vertices with  $r$  edges.

For every graph  $H$ , define  $\mathcal{I}(H)$  to be the set of graphs containing  $H$  as an immersion.  $\mathcal{I}(H)$  does not have the edge-variant of the Erdős-Pósa property for every graph  $H$ . The necessary conditions for graph  $H$  in which  $\mathcal{TM}(H)$  has the Erdős-Pósa property mentioned in [4] are necessary for graph  $H$  in which  $\mathcal{I}(H)$  has the edge-variant of the Erdős-Pósa property. On the other hand, even though a family of graphs does not have the edge-variant of the Erdős-Pósa property, it probably has if we restrict the host graphs to a smaller class of graphs. For example, the set of odd cycles does not have the edge-variant of the Erdős-Pósa property, but Kawarabayashi and Kobayashi [3] proved that it has the edge-variant of the Erdős-Pósa property

in 4-edge-connected graphs. We address the same direction in this paper and prove that for every graph  $H$ ,  $\mathcal{I}(H)$  has the edge-variant of the Erdős-Pósa property in 4-edge-connected graphs. In other words, we prove the following theorem.

**Theorem 1.1.** *For every graph  $H$ , there exists a function  $f : \mathbb{N} \rightarrow \mathbb{N}$  such that for every 4-edge-connected graph  $G$  and for every positive integer  $k$ , either  $G$  contains  $k$  edge-disjoint subgraphs each containing  $H$  as an immersion, or there exists  $Z \subseteq E(G)$  with  $|Z| \leq f(k)$  such that  $G - Z$  has no  $H$ -immersion.*

The necessary conditions for which  $\mathcal{TM}(H)$  has the Erdős-Pósa property mentioned in [4] provide necessary conditions for which  $\mathcal{I}(H)$  has the edge-variant Erdős-Pósa property and show that the 4-edge-connectivity cannot be replaced by the 3-edge-connectivity.

We also consider the following version of the half-integral packing problem in this paper. For every graph  $H$ , an  $H$ -half-integral immersion in  $G$  is a pair of functions  $(\pi_V, \pi_E)$  such that the following hold.

- $\pi_V$  is an injection from  $V(H)$  to  $V(G)$ .
- $\pi_E$  maps every loop with the end  $v$  of  $H$  to a cycle in  $G$  containing  $\pi_V(v)$  and maps every non-loop edge  $e$  with ends  $u, v$  of  $G$  to a path in  $G$  from  $\pi_V(u)$  to  $\pi_V(v)$ .
- For every edge  $e$  of  $G$ , there exist at most two edges  $e_1, e_2$  of  $H$  such that  $e \in \pi_E(e_1)$  and  $e \in \pi_E(e_2)$ .

The following theorem shows that the 4-edge-connectivity can be dropped if we consider the following version of half-integral packing of half-integral immersions.

**Theorem 1.2.** *For every graph  $H$ , there exists a function  $f : \mathbb{N} \rightarrow \mathbb{N}$  such that for every graph  $G$  and for every positive integer  $k$ , either*

1.  *$G$  contains  $k$   $H$ -half-integral immersions  $(\pi_V^{(1)}, \pi_E^{(1)}), \dots, (\pi_V^{(k)}, \pi_E^{(k)})$  such that for every edge  $e$  of  $G$ , there exist at most two pairs  $(i, e')$  with  $1 \leq i \leq k$  and  $e' \in E(H)$  such that  $e \in \pi_E^{(i)}(e')$ , or*
2. *there exists  $Z \subseteq E(G)$  with  $|Z| \leq f(k)$  such that  $G - Z$  has no  $H$ -half-integral immersion.*

Now we sketch the proof of Theorem 1.1 and describe the organization of this paper. Tangle is one of the important notions in Robertson and Seymour's Graph Minors series. It defines an orientation for each separation of small order in a graph and has been proven useful in dealing with problems related with Erdős-Pósa property. In Section 2, we develop a similar machinery called "edge-tangle" but addressing edge-cuts. This is one of the cornerstone of this paper. In Section 3, we prove a structure theorem for excluding a fixed graph as an immersion when an edge-tangle "grasps" a set of many pairwise edge-disjoint but pairwise intersecting subgraphs. It will provide the first step for the proof of Theorem 1.1, as every graph with no  $k$  edge-disjoint  $H$ -immersions has no certain graphs as immersions as well. In Section 4, we prove that every "sufficiently large" 4-edge-connected graph has an edge-tangle satisfying the property mentioned in Section 3. Finally, we prove Theorems 1.1 and 1.2 in Section 5.

We define some notation to conclude this section. Given a subset  $X$  of the vertex-set  $V(G)$  of a graph  $G$ , the subgraph of  $G$  induced by  $X$  is denoted by  $G[X]$ , and the set of vertices that are not in  $X$  but adjacent to some vertices in  $X$  is denoted by  $N_G(X)$ . When  $X = \{v\}$ , we write  $N_G(\{v\})$  as  $N_G(v)$  for simplicity. We define  $N_G[X] = N_G(X) \cup X$  and  $N_G[v] = N_G(v) \cup \{v\}$ . A graph is *simple* if it does not contain parallel edges and loops. The *line graph* of a graph  $G$ , denoted by  $L(G)$ , is the simple graph with  $V(L(G)) = E(G)$ , and every pair of vertices  $x, y \in V(L(G))$  are adjacent in  $L(G)$  if and only if  $x, y$  are two edges having a common end in  $G$ . For every  $v \in V(G)$ , define  $\text{cl}(v)$  to be the clique in  $L(G)$  consisting of the edges of  $G$  incident with  $v$ . Given a function  $f$  and a subset  $S$  of its domain, we define  $f(S) = \{f(x) : x \in S\}$ . The *degree* of a vertex  $v$  in a graph  $G$ , denoted by  $\deg_G(v)$ , is the number of edges of  $G$  incident with  $v$ , where each loop is counted twice. If  $G$  is a graph and  $Y \subseteq V(G)$ , then  $G - Y$  is the graph  $G[V(G) - Y]$ ; if  $Y \subseteq E(G)$ , then  $G - Y$  is the graph with  $V(G - Y) = V(G)$  and  $E(G - Y) = E(G) - Y$ . For a positive integer  $k$ , a graph  $G$  is  *$k$ -edge-connected* if  $G$  contains at least two vertices and  $G - F$  is connected for every  $F \subseteq E(G)$  with  $|F| < k$ . For every positive integer  $n$ , we denote the set  $\{1, 2, \dots, n\}$  by  $[n]$  for short.

## 2 Tangles and edge-tangles

A *separation* of a graph  $G$  is an ordered pair  $(A, B)$  of edge-disjoint subgraphs of  $G$  with  $A \cup B = G$ , and the *order* of  $(A, B)$  is  $|V(A) \cap V(B)|$ . A

*tangle*  $\mathcal{T}$  in  $G$  of *order*  $\theta$  is a set of separations of  $G$  of order less than  $\theta$  such that

- (T1) for every separation  $(A, B)$  of  $G$  of order less than  $\theta$ , either  $(A, B) \in \mathcal{T}$  or  $(B, A) \in \mathcal{T}$ ;
- (T2) if  $(A_1, B_1), (A_2, B_2), (A_3, B_3) \in \mathcal{T}$ , then  $A_1 \cup A_2 \cup A_3 \neq G$ ;
- (T3) if  $(A, B) \in \mathcal{T}$ , then  $V(A) \neq V(G)$ .

The notion of tangles was first defined by Roberson and Seymour in [7]. We call (T1), (T2) and (T3) the *first*, *second* and *third tangle axioms*, respectively. Note that (T2) implies that if  $(A, B) \in \mathcal{T}$ , then  $(B, A) \notin \mathcal{T}$ .

A separation  $(A, B)$  of  $G$  is *normalized* if every vertex  $v \in V(A) \cap V(B)$  is adjacent to a vertex of  $A - V(B)$  and adjacent to a vertex in  $B - V(A)$ . The *normalization* of a separation  $(A, B)$  of a graph  $G$  is the separation obtained from  $(A, B)$  by first removing non-isolated vertices  $v \in V(A) \cap V(B)$  with  $N_A(v) \subseteq V(B)$  from  $A$  and put all edges of  $G$  incident with  $v$  into  $B$ , and second moving all edges in  $B$  whose both ends in  $V(A) \cap V(B)$  into  $A$ , and finally removing all isolated vertices of  $B$  from  $B$  and put them into  $A$ . It is clear that the normalization of any separation is normalized, and the order of the normalization of  $(A, B)$  is no more than the order of  $(A, B)$ .

**Lemma 2.1.** *Let  $\theta$  be a positive integer and  $\mathcal{T}$  a tangle of order  $\theta$  in a graph  $G$ . Then for every separation  $(A, B)$  of  $G$  of order less than  $\theta$ ,  $(A, B) \in \mathcal{T}$  if and only if its normalization belongs to  $\mathcal{T}$ .*

**Proof.** Let  $(A, B)$  be a separation of  $G$  of order less than  $\theta$ , and let  $(A', B')$  be its normalization. By (T1) and (T3),  $(G[V(A) \cap V(B)], G - E(G[V(A) \cap V(B)])) \in \mathcal{T}$ . Let  $X$  be the set of isolated vertices of  $G$ . For every  $x \in X$ ,  $(G[\{x\}], G) \in \mathcal{T}$  by (T3), and hence  $(G[\{x\}], G - \{x\}) \in \mathcal{T}$  by (T2). By repeatedly applying (T2),  $(G[X], G - X) \in \mathcal{T}$ . So  $(G[V(A) \cap V(B)] \cup G[X], G - E(G[V(A) \cap V(B)]) - X) \in \mathcal{T}$  by (T1) and (T2).

Suppose that  $(A, B) \in \mathcal{T}$  but  $(A', B') \notin \mathcal{T}$ . Since the order of  $(B', A')$  is no more than the order of  $(A', B')$ ,  $(B', A') \in \mathcal{T}$  by (T1). Since  $(A, B)$ ,  $(B', A')$  and  $(G[V(A) \cap V(B)] \cup G[X], G - E(G[V(A) \cap V(B)]) - X)$  belong to  $\mathcal{T}$ , (T2) implies that  $A \cup B' \cup G[V(A) \cap V(B)] \cup G[X] \neq G$ . In other words,  $A \cup B' \cup G[V(A) \cap V(B)]$  does not contain all edges of  $G$ . Let  $e \in E(G) - E(A \cup B' \cup G[V(A) \cap V(B)])$ , so  $e \in E(B) - E(B')$  and  $e$  has an end not in  $V(A) \cap V(B)$ . But  $e \in E(B) - E(B')$  implies that  $e$  has both ends in

$V(A) \cap V(B)$  by the definition of the normalization, a contradiction. This proves that  $(A, B) \in \mathcal{T}$  implies that  $(A', B') \in \mathcal{T}$ .

Suppose that  $(A, B) \notin \mathcal{T}$  but  $(A', B') \in \mathcal{T}$ . Then  $(B, A) \in \mathcal{T}$  by (T1). Since  $(A', B')$ ,  $(B, A)$  and  $(G[V(A) \cap V(B)] \cup G[X], G - E(G[V(A) \cap V(B)]) - X)$  belong to  $\mathcal{T}$ ,  $A' \cup B \cup G[V(A) \cap V(B)] \cup G[X] \neq G$  by (T2). Hence, some edge  $e'$  of  $G$  is not in  $A' \cup B \cup G[V(A) \cap V(B)]$ . So  $e' \in E(B') - E(B)$  and  $e'$  has some end not in  $V(A) \cap V(B)$ . By the definition of the normalization,  $e'$  is incident with a vertex  $v$  of  $G$  with  $N_A(v) \subseteq V(B)$ . But this implies that  $e'$  has both ends in  $V(A) \cap V(B)$ , a contradiction. This proves that  $(A', B') \in \mathcal{T}$  implies that  $(A, B) \in \mathcal{T}$ . ■

An *edge-cut* of a graph  $G$  is an ordered partition  $[A, B]$  of  $V(G)$ , where some of  $A$  and  $B$  is allowed to be empty. The *order* of an edge-cut  $[A, B]$ , denoted by  $|[A, B]|$ , is the number of edges with one end in  $A$  and one end in  $B$ . For an edge  $e$  of  $G$ , we write  $e \in [A, B]$  if  $e$  has one end in  $A$  and one end in  $B$ .

The *partner* of a normalized separation  $(A, B)$  of the line graph  $L(G)$  of  $G$  is the edge-cut  $[A', B']$  of  $G$  satisfying that  $A'$  is the union of the set of isolated vertices of  $G$  and the set  $\{v \in V(G) : V(\text{cl}(v)) \subseteq V(A)\}$ , and  $B' = \{v \in V(G) : V(\text{cl}(v)) \subseteq V(B)\}$ . Note that the partner of  $(A, B)$  is well-defined since  $(A, B)$  is normalized.

**Lemma 2.2.** *Let  $G$  be a connected graph, and let  $(A, B)$  be a separation of  $L(G)$ . If  $(A, B)$  is normalized, then the order of  $(A, B)$  equals the order of its partner  $[A', B']$ .*

**Proof.** Let  $e \in [A', B']$  with ends  $u, v$ , where  $u \in A'$  and  $v \in B'$ . So  $V(\text{cl}(u)) \subseteq V(A)$  and  $V(\text{cl}(v)) \subseteq V(B)$ . Hence,  $e \in V(\text{cl}(u)) \cap V(\text{cl}(v)) \subseteq V(A) \cap V(B)$ . This implies that the order of  $(A, B)$  is at least the order of  $[A', B']$ .

On the other hand, let  $e \in V(A) \cap V(B)$ . Since  $(A, B)$  is normalized,  $e$  is adjacent to a vertex  $e_A$  of  $L(G)$  in  $V(A) - V(B)$  and a vertex  $e_B$  of  $L(G)$  in  $V(B) - V(A)$ . So  $e$  and  $e_A$  have a common end  $x$  in  $G$ , and  $e$  and  $e_B$  have a common end  $y$  of  $G$ . Since  $e_A \notin V(B)$ ,  $V(\text{cl}(x)) \subseteq V(A)$  and hence  $x \in A'$ . Similarly,  $V(\text{cl}(y)) \subseteq V(B)$  and  $y \in B'$ . So  $x \neq y$  and they are the ends of  $e$ . This proves that  $e \in [A', B']$  and the order of  $(A, B)$  is at most the order of  $[A', B']$ . ■

An *edge-tangle*  $\mathcal{E}$  in a graph  $G$  of order  $\theta$  is a set of edge-cuts of  $G$  of order less than  $\theta$  such that the following hold.

- (E1) For every edge-cut  $[A, B]$  of  $G$  of order less than  $\theta$ , either  $[A, B] \in \mathcal{E}$  or  $[B, A] \in \mathcal{E}$ ;
- (E2) If  $[A_1, B_1], [A_2, B_2], [A_3, B_3] \in \mathcal{E}$ , then  $B_1 \cap B_2 \cap B_3 \neq \emptyset$ .
- (E3) If  $[A, B] \in \mathcal{E}$ , then  $G$  has at least  $\theta$  edges incident with vertices in  $B$ .

We call (E1), (E2) and (E3) the *first*, *second* and *third edge-tangle axioms*, respectively. Note that if an edge-tangle  $\mathcal{E}$  of order  $\theta \geq 1$  in  $G$  exists, then  $[\emptyset, V(G)] \in \mathcal{E}$  by (E1) and (E2), so  $|E(G)| \geq \theta$  by (E3). Furthermore, for every  $[A, B] \in \mathcal{E}$ , there exists an edge of  $G$  with both ends in  $B$  by (E3).

Given an edge-tangle  $\mathcal{E}$  of order  $\theta$  in  $G$ , the *conjugate*  $\overline{\mathcal{E}}$  of  $\mathcal{E}$  is the set of separations of  $L(G)$  of order less than  $\theta/3$  of such that  $(A, B) \in \overline{\mathcal{E}}$  if and only if the partner of the normalization of  $(A, B)$  is in  $\mathcal{E}$ .

**Lemma 2.3.** *Let  $\theta$  be a positive integer and  $G$  a graph. If  $\mathcal{E}$  is an edge-tangle of order  $3\theta - 2$  of  $G$ , then  $\overline{\mathcal{E}}$  is a tangle of order  $\theta$  in  $L(G)$ .*

**Proof.** We shall prove that  $\overline{\mathcal{E}}$  satisfies tangle axioms (T1), (T2) and (T3). Note that  $|E(G)| \geq 3\theta - 2$  since  $G$  has an edge-tangle of order  $3\theta - 2$ .

First, we show that  $\overline{\mathcal{E}}$  satisfies (T1). Let  $(A, B)$  is a separation of  $L(G)$  of order less than  $\theta$ . Let  $(A_1, B_1)$  and  $(B_2, A_2)$  be the normalizations of  $(A, B)$  and  $(B, A)$ , respectively. And let  $[A'_1, B'_1]$  and  $[B'_2, A'_2]$  be the partners of  $(A_1, B_1)$  and  $(B_2, A_2)$ , respectively. If any of  $[A'_1, B'_1]$  and  $[B'_2, A'_2]$  is in  $\mathcal{E}$ , then  $(A, B)$  or  $(B, A)$  is in  $\overline{\mathcal{E}}$ , so we may assume that none of  $[A'_1, B'_1]$  and  $[B'_2, A'_2]$  is in  $\mathcal{E}$ . By Lemma 2.2, the order of  $[A'_1, B'_1]$  and  $[B'_2, A'_2]$  are less than  $\theta$ , so  $[B'_1, A'_1]$  and  $[A'_2, B'_2]$  are in  $\mathcal{E}$  by (E1). Note that if  $v \in A'_1 \cap B'_2$ , then  $V(\text{cl}(v)) \subseteq V(A_1) \cap V(B_2)$ , so  $\text{cl}(v)$  is not an isolated vertex in  $L(G)$  and  $V(\text{cl}(v)) \subseteq V(A) \cap V(B)$ . Therefore, the number of edges of  $G$  incident with some vertex in  $A'_1 \cap B'_2$  is at most  $|\bigcup_{v \in A'_1 \cap B'_2} V(\text{cl}(v))| \leq |V(A) \cap V(B)| < \theta$ . Hence, (E1) implies that either  $[A'_1 \cap B'_2, B'_1 \cup A'_2] \in \mathcal{E}$  or  $[B'_1 \cup A'_2, A'_1 \cap B'_2] \in \mathcal{E}$ . But (E3) excludes the latter case, so  $[A'_1 \cap B'_2, B'_1 \cup A'_2] \in \mathcal{E}$ . However,  $[B'_1, A'_1]$ ,  $[A'_2, B'_2]$  and  $[A'_1 \cap B'_2, B'_1 \cup A'_2]$  belong to  $\mathcal{E}$ , but  $A'_1 \cap B'_2 \cap (B'_1 \cup A'_2) = \emptyset$ , contradicting (E2). This proves that  $\overline{\mathcal{E}}$  satisfies (T1).

Next, we show that  $\overline{\mathcal{E}}$  satisfies (T3). Suppose that  $(A, B) \in \overline{\mathcal{E}}$  with  $V(A) = V(L(G))$ . So the partner  $[A', B']$  of the normalization of  $(A, B)$  is in  $\mathcal{E}$ . Note that  $\bigcup_{v \in B'} V(\text{cl}(v)) \subseteq V(B) = V(A) \cap V(B)$ , so the number of edges incident with vertices in  $B'$  is at most  $|V(A) \cap V(B)| \leq (3\theta - 3)/3 < \theta$ , a contradiction. Consequently,  $\overline{\mathcal{E}}$  satisfies (T3).



Now we suppose that  $\overline{\mathcal{E}}$  does not satisfy (T2). So there exist separations  $(A_1, B_1)$ ,  $(A_2, B_2)$ ,  $(A_3, B_3)$  in  $\overline{\mathcal{E}}$  such that  $A_1 \cup A_2 \cup A_3 = L(G)$ . For each  $1 \leq i \leq 3$ , let  $(A_i^*, B_i^*)$  be the normalization of  $(A_i, B_i)$ , and let  $[A'_i, B'_i]$  be the partner of  $(A_i^*, B_i^*)$ . By the definition of  $\overline{\mathcal{E}}$ ,  $[A'_i, B'_i] \in \mathcal{E}$  for  $1 \leq i \leq 3$ . The number of edges incident with vertices in  $\bigcap_{i=1}^3 B'_i$  is at most  $|\bigcup_{v \in \bigcap_{i=1}^3 B'_i} V(\text{cl}(v))| \leq |\bigcap_{i=1}^3 V(B_i^*)| \leq |\bigcap_{i=1}^3 V(B_i)|$ . However,  $\bigcap_{i=1}^3 V(B_i) \subseteq \bigcup_{i=1}^3 V(A_i \cap B_i)$ , as  $A_1 \cup A_2 \cup A_3 = G$ . So the number of edges incident with vertices in  $\bigcap_{i=1}^3 B'_i$  is at most  $|\bigcup_{i=1}^3 V(A_i \cap B_i)| \leq 3(\theta - 1)$ . On the other hand,  $|[A'_1 \cup A'_2 \cup A'_3, B'_1 \cap B'_2 \cap B'_3]| \leq \sum_{i=1}^3 |[A'_i, B'_i]| \leq 3(\theta - 1)$ . So  $[A'_1 \cup A'_2 \cup A'_3, B'_1 \cap B'_2 \cap B'_3]$  or  $[B'_1 \cap B'_2 \cap B'_3, A'_1 \cup A'_2 \cup A'_3]$  is in  $\mathcal{E}$  by (E1). Since  $[A'_i, B'_i] \in \mathcal{E}$  for  $1 \leq i \leq 3$ , (E2) implies that  $[A'_1 \cup A'_2 \cup A'_3, B'_1 \cap B'_2 \cap B'_3] \in \mathcal{E}$ . Therefore, there are at least  $3\theta - 2$  edges incident with vertices in  $B'_1 \cap B'_2 \cap B'_3$ , a contradiction. This proves that  $\overline{\mathcal{E}}$  satisfies (T3). Consequently,  $\overline{\mathcal{E}}$  is a tangle of order  $\theta$  in  $L(G)$ . ■

Let  $G$  be a graph and  $\mathcal{E}$  a collection of edge-cuts of  $G$  of order less than a positive number  $\theta$ , and let  $X \subseteq E(G)$ . Define  $\mathcal{E} - X$  to be the set of edge-cuts of  $G - X$  of order less than  $\theta - |X|$  such that  $[A, B] \in \mathcal{E} - X$  if and only if  $[A, B] \in \mathcal{E}$ .

**Lemma 2.4.** *Let  $G$  be a graph and  $\theta$  a positive integer. If  $\mathcal{E}$  is an edge-tangle in  $G$  of order  $\theta$  and  $X$  is a subset of  $E(G)$  with  $|X| < \theta$ , then  $\mathcal{E} - X$  is an edge-tangle in  $G - X$  of order  $\theta - |X|$ .*

**Proof.** If  $[A, B]$  is an edge-cut of order less than  $\theta - |X|$  in  $G - X$ , then  $[A, B]$  is an edge-cut in  $G$  of order less than  $\theta$ . So  $\mathcal{E} - X$  satisfies (E1). Since  $\mathcal{E}$  satisfies (E2),  $B_1 \cap B_2 \cap B_3 \neq \emptyset$ , for every three edge-cuts  $[A_1, B_1], [A_2, B_2], [A_3, B_3] \in \mathcal{E} - X$ . So  $\mathcal{E} - X$  satisfies (E2). If  $[A, B] \in \mathcal{E} - X$ , then  $[A, B] \in \mathcal{E}$ , so  $G$  contains at least  $\theta$  edges incident with vertices in  $B$ . Hence,  $G - X$  contains at least  $\theta - |X|$  edges incident with some vertices in  $B$ . This proves that  $\mathcal{E} - X$  is an edge-tangle of order  $\theta - |X|$ . ■

Let  $\mathcal{T}$  be a tangle in a graph  $G$ . We say that a subset  $X$  of  $V(G)$  is *free* with respect to  $\mathcal{T}$  if there does not exist  $(A, B) \in \mathcal{T}$  of order less than  $|X|$  such that  $X \subseteq V(A)$ . Let  $\mathcal{E}$  be an edge-tangle in a graph  $G$ . We say that a subset  $Y$  of  $E(G)$  is *free* with respect to  $\mathcal{E}$  if there exist no  $Z \subseteq Y$  and  $[A, B] \in \mathcal{E} - Z$  of order less than  $|Y - Z|$  such that every edge in  $Y - Z$  has both ends in  $A$ .

**Lemma 2.5.** *Let  $\mathcal{E}$  be an edge-tangle in a graph  $G$ , and let  $\overline{\mathcal{E}}$  be the conjugate of  $\mathcal{E}$ . Let  $X$  be a subset of  $E(G)$  and let  $Y$  be the subset of  $V(L(G))$  corresponding to  $X$ . If  $X$  is free with respect to  $\mathcal{E}$ , then  $Y$  is free with respect to  $\overline{\mathcal{E}}$ .*

**Proof.** Suppose that  $Y$  is not free with respect to  $\overline{\mathcal{E}}$ . So there exists a separation  $(A, B) \in \overline{\mathcal{E}}$  of  $L(G)$  of order less than  $|Y|$  such that  $Y \subseteq V(A)$ . We may assume that the order of  $(A, B)$  is as small as possible, and subject to that,  $V(B)$  is as small as possible. So every vertex in  $V(A) \cap V(B) - Y$  is adjacent to a vertex in  $V(A) - V(B)$  and adjacent to a vertex in  $V(B) - V(A)$ ; every vertex in  $V(A) \cap V(B) \cap Y$  is adjacent to a vertex in  $V(B) - V(A)$ . Furthermore,  $B$  has no isolated vertices. Let  $(A^*, B^*)$  be the normalization of  $(A, B)$ . So  $V(B^*) = V(B)$  and  $|V(A^*) \cap V(B^*) - X| = |V(A) \cap V(B)| - |V(A) \cap V(B) \cap X|$ .

Let  $[A', B']$  be the partner of  $(A^*, B^*)$ . Let  $W$  be the subset of  $Y$  corresponding to  $V(A) \cap V(B) \cap X$ . Every edge  $e$  in  $X - W$  has both ends in  $A'$  since it corresponds to a vertex in  $V(A) - V(B)$ . And the order of  $[A', B']$  in  $G - W$  equals  $|V(A^*) \cap V(B^*) - X| = |V(A) \cap V(B)| - |W| < |X| - |W|$ . So  $X$  is not free with respect to  $\mathcal{E}$ , a contradiction. ■

**Lemma 2.6.** *Let  $\mathcal{E}$  be an edge-tangle in a graph  $G$ , and let  $\overline{\mathcal{E}}$  be the conjugate of  $\mathcal{E}$ . Let  $X$  be a subset of  $E(G)$  and let  $Y$  be the subset of  $V(L(G))$  corresponding to  $X$ . Denote the order of  $\overline{\mathcal{E}}$  by  $\theta$ . If  $Y$  is free with respect to  $\overline{\mathcal{E}}$  and  $|Y| \leq \theta$ , then  $X$  is free with respect to  $\mathcal{E}$ .*

**Proof.** Suppose that  $X$  is not free with respect to  $\mathcal{E}$ . So there exists  $W \subseteq X$  and  $[A, B] \in \mathcal{E} - W$  of order less than  $|X - W|$  such that every edge in  $X - W$  has both ends in  $A$ . Define  $B'$  to be a subgraph of  $L(G - W)$  such that  $V(B')$  corresponds to the edges of  $G$  incident with vertices of  $B$ . Define  $(A', B')$  to be a separation of  $L(G - W)$  such that  $V(A') \cap V(B')$  corresponds to the edges of  $G$  between  $A$  and  $B$ . Note that the order of  $(A', B')$  is at most the order of  $[A, B]$  in  $G - W$ . So  $|V(A') \cap V(B')| < |X - W|$ .

Let  $W'$  be the set of vertices of  $L(G)$  corresponding to  $W$ . Let  $(A^*, B^*)$  be a separation of  $L(G)$  with  $V(A^*) = V(A') \cup W'$  and  $V(B^*) = V(B') \cup W'$ . So  $Y \subseteq V(A^*)$  and  $|V(A^*) \cap V(B^*)| < |X| \leq \theta$ . By (T1),  $(A^*, B^*) \in \overline{\mathcal{E}}$  or  $(B^*, A^*) \in \overline{\mathcal{E}}$ .

Let  $(A'', B'')$  be the normalization of  $(A^*, B^*)$ . Since every edge in  $X - W$  has both ends in  $A$ ,  $V(A^*) - V(B^*) \neq \emptyset$  and hence  $V(B'') \neq V(L(G))$ . Since

every vertex in  $V(A') \cap V(B')$  has a neighbor in  $V(A') - V(B')$  and a neighbor in  $V(B') - V(A')$ ,  $V(A^*) \cap V(B^*) - W' \subseteq V(A'') \cap V(B'')$ .

Let  $[C, D]$  be the partner of  $(A'', B'')$  in  $G$ . Then  $[C, D] = [A, B]$ , so  $(A^*, B^*) \in \overline{\mathcal{E}}$ . But  $Y \subseteq V(A^*)$  and  $|V(A^*) \cap V(B^*)| < |X|$ , so  $Y$  is not free with respect to  $\overline{\mathcal{E}}$ , a contradiction. This proves that  $X$  is free with respect to  $\mathcal{E}$ . ■

The following lemma provides a way to obtain an edge-tangle from an immersion.

**Lemma 2.7.** *Let  $H$  be a graph and  $\mathcal{E}'$  an edge-tangle of order  $\theta$  in  $H$ . Let  $G$  be a graph that contains an  $H$ -immersion  $(\pi_V, \pi_E)$ . If  $\mathcal{E}$  is the set of all edge-cuts  $[A, B]$  of  $G$  of order less than  $\theta$  such that there exists  $[A', B'] \in \mathcal{E}'$  with  $\pi_V(A') = A \cap \pi_V(V(H))$ , then  $\mathcal{E}$  is an edge-tangle of order  $\theta$  in  $G$ .*

**Proof.** We shall show that  $\mathcal{E}$  satisfies the edge-tangle axioms (E1), (E2) and (E3). Let  $[A, B]$  be an edge-cut of  $G$  of order less than  $\theta$ .  $A \cap \pi_V(V(H))$  and  $B \cap \pi_V(V(H))$  are two subsets of  $\pi_V(V(H))$  such that their union is  $\pi_V(V(H))$ , so there exists an edge-cut  $[A', B']$  of  $H$  such that  $\pi_V(A') = A \cap \pi_V(V(H))$  and  $\pi_V(B') = B \cap \pi_V(V(H))$ . Since  $(\pi_V, \pi_E)$  is an  $H$ -immersion, there are at least  $|[A', B']|$  edge-disjoint paths in  $G$  from  $A \cap \pi_V(V(H))$  to  $B \cap \pi_V(V(H))$ . So  $|[A', B']| \leq |[A, B]| < \theta$ . Hence, one of  $[A', B']$  and  $[B', A']$  is in  $\mathcal{E}'$ , and hence one of  $[A, B]$  and  $[B, A]$  is in  $\mathcal{E}$ . So  $\mathcal{E}$  satisfies (E1).

For each  $i$  with  $1 \leq i \leq 3$ , let  $[A_i, B_i] \in \mathcal{E}$  be an edge-cut of  $G$ , and let  $[A'_i, B'_i] \in \mathcal{E}'$  such that  $\pi_V(B'_i) = B_i \cap \pi_V(V(H))$ . Since  $\mathcal{E}'$  is an edge-tangle in  $H$ ,  $B'_1 \cap B'_2 \cap B'_3$  contains a vertex  $v$  of  $H$ . So  $\pi_V(v) \in B_1 \cap B_2 \cap B_3$ . This proves that  $\mathcal{E}$  satisfies (E2).

Finally, we prove that  $\mathcal{E}$  satisfies (E3). Let  $[A, B] \in \mathcal{E}$  and let  $[A', B'] \in \mathcal{E}'$  such that  $\pi_V(B') = B \cap \pi_V(V(H))$ . Since  $\mathcal{E}'$  satisfies (E3),  $H$  contains at least  $\theta$  edges incident with vertices in  $B'$ . So there are at least  $\theta$  edge-disjoint subgraphs of  $G$  each containing a vertex in  $\pi_V(B') \subseteq B$ . Therefore,  $G$  contains at least  $\theta$  edges incident with vertices in  $B$ . Consequently,  $\mathcal{E}$  is an edge-tangle in  $G$ . ■

We call the edge-tangle  $\mathcal{E}$  defined in Lemma 2.7 the *edge-tangle induced by the  $H$ -immersion  $(\pi_V, \pi_E)$* .

The  $m \times n$  wall is the simple graph with vertex-set  $\{(i, j) : 1 \leq i \leq n, 1 \leq j \leq m\}$  and edge-set  $\{(i, j)(i+1, j) : 1 \leq i \leq n-1, 1 \leq j \leq m\} \cup \{(2a-1, 2b-1)(2a-1, 2b) : 1 \leq a \leq \lceil n/2 \rceil, 1 \leq b \leq \lfloor m/2 \rfloor\} \cup \{(2a, 2b)(2a, 2b+1) :$

$1 \leq a \leq \lfloor n/2 \rfloor, 1 \leq b \leq \lfloor (m-1)/2 \rfloor$ . The  $i$ -th row of the  $m \times n$  wall is the subgraph induced by  $\{(x, i) : 1 \leq x \leq n\}$ . The  $k$ -th column of the  $m \times n$  wall is the subgraph induced by  $\{(x, y) : 2k-1 \leq x \leq \min\{2k, n\}, 1 \leq y \leq m\}$ . Hence, the  $m \times n$  wall contains  $m$  rows and  $\lceil n/2 \rceil$  columns.

**Lemma 2.8.** *Let  $r$  and  $\theta$  be positive integers with  $\theta \leq r$ . Let  $G$  be the  $2r \times r$  wall. If  $[A, B]$  is an edge-cut of order less than  $\theta$  of  $G$ , then*

1. *exactly one of  $A$  and  $B$  contains all vertices of a column of  $G$ ,*
2. *exactly one of  $A$  and  $B$  contains all vertices of a row of  $G$ , and*
3.  *$A$  contains all vertices of a column if and only if  $A$  contains all vertices of a row.*

**Proof.** Note that  $G$  has  $r$  rows and  $r$  columns. Suppose that  $A$  contains all vertices of a column and  $B$  contains all vertices of another column. Then every row must contain an edge in  $[A, B]$ , so the order of  $[A, B]$  is at least  $r$ , a contradiction. Suppose that none of  $A$  and  $B$  contains all vertices of a column. Then every column must contain an edge in  $[A, B]$ , so the order of  $[A, B]$  is at least  $r$ , a contradiction. So exactly one of  $A$  and  $B$  contains all vertices of a column. Similarly, exactly one of  $A$  and  $B$  contains all vertices of a row. Furthermore, if  $A$  contains all vertices of a column, then  $B$  cannot contain all vertices of a row, so  $A$  contains all vertices of a row as well. Similarly, if  $B$  contains all vertices of a column, then  $B$  contains all vertices of a row. ■

**Lemma 2.9.** *Let  $r$  and  $\theta$  be positive integers. Let  $G$  be the  $2r \times r$  wall. Let  $\mathcal{E}$  be the set of all edge-cuts  $[A, B]$  of order less than  $\theta$  of  $G$  satisfying that  $B$  contains all vertices of a column of  $G$ . If  $r \geq 2\theta$ , then  $\mathcal{E}$  is an edge-tangle of  $G$  of order  $\theta$ .*

**Proof.** Let  $[A, B]$  be an edge-cut of  $G$  of order less than  $\theta$ . By Lemma 2.8,  $[A, B]$  or  $[B, A]$  is in  $\mathcal{E}$ . Hence,  $\mathcal{E}$  satisfies the first edge-tangle axiom.

Let  $[A_i, B_i] \in \mathcal{E}$  be edge-cuts of  $G$  for  $1 \leq i \leq 3$ . Let  $c_i$  be a column of  $G$  contained in  $B_i$  for  $1 \leq i \leq 3$ . Suppose that  $B_1 \cap B_2 \cap B_3 = \emptyset$ . If  $c_1 = c_2$ , then  $A_3$  contains  $c_1$ , a contradiction. So by symmetry, we may assume that  $c_1, c_2, c_3$  are pairwise distinct. Since  $A_2$  or  $A_3$ , say  $A_2$ , contains at least one half vertices of  $c_1$ ,  $A_2$  contains vertices of at least  $r/2$  rows. But  $B_2$  contains  $c_2$ , so there are at least  $r/2$  edges with one end in  $A_2$  and one end in  $B_2$ .

Therefore,  $[A_2, B_2]$  has order at least  $r/2 \geq \theta$ , a contradiction. Hence,  $\mathcal{E}$  satisfies (E2).

For every  $[A, B] \in \mathcal{E}$ ,  $B$  contains a column in  $G$ , so there are at least  $r \geq \theta$  edges in  $G$  incident with some vertices in  $B$ . This proves that  $\mathcal{E}$  is an edge-tangle. ■

The following two lemmas will be used in Section 3.

**Lemma 2.10.** *Let  $G$  be a graph and  $\mathcal{E}$  an edge-tangle in  $G$ . Let  $p$  be a positive integer and let  $[A_1, B_1], \dots, [A_p, B_p] \in \mathcal{E}$ . For each  $i$  with  $1 \leq i \leq p$ , let  $X_i$  be the set of edges of  $G$  between  $A_i$  and  $B_i$ . Assume that for every  $i$  with  $1 \leq i \leq p$  and for every  $v \in A_i$ , there exists a path in  $G[A_i]$  from  $v$  to an end of an edge in  $X_i$ . Assume the order of  $\mathcal{E}$  is greater than  $|\bigcup_{i=1}^p X_i|$ . If  $\bigcup_{i=1}^p X_i$  is free with respect to  $\mathcal{E}$  and  $X_i \cap X_j = \emptyset$  for every pair of distinct  $i, j$ , then  $A_i \cap A_j = \emptyset$  for every pair of distinct  $i, j$ .*

**Proof.** There is nothing to prove if  $p = 1$ , so we may assume that  $p \geq 2$ . First, we suppose that there exists an edge  $e \in X_1$  such that the both ends of  $e$  are in  $A_2$ . Let  $Z = (\bigcup_{i=1}^p X_i) - e$ . Since the order of  $\mathcal{E}$  is greater than  $|\bigcup_{i=1}^p X_i|$ ,  $[A_2, B_2] \in \mathcal{E} - Z$  has order zero in  $G - Z$ . But the both ends of the unique member  $e$  of  $\bigcup_{i=1}^p X_i - Z$  are in  $A_2$ . So  $\bigcup_{i=1}^p X_i$  is not free with respect to  $\mathcal{E}$ , a contradiction. Hence, no edge in  $X_1$  has both ends in  $A_2$ . Similarly, for every pair of distinct  $i, j$ , no edge in  $X_i$  has both ends in  $X_j$ .

Now we suppose that there exist distinct  $i, j$  such that  $A_i \cap A_j \neq \emptyset$ . Let  $v \in A_i \cap A_j$ . Let  $P_i$  be a path in  $G[A_i]$  from  $v$  to an end of an edge  $e'$  in  $X_i$ . Since  $X_i$  is disjoint from  $X_j$  and some end of  $e'$  is in  $B_j$ ,  $e'$  has both ends in  $B_j$ . So  $P_i$  intersects  $X_j$ . But every edge in  $X_j \cap E(P_i)$  has both ends in  $A_i$ , a contradiction. This proves the lemma. ■

**Lemma 2.11.** *Let  $\xi$  be a positive integer. Let  $G$  be a graph and  $\mathcal{E}$  an edge-tangle in  $G$  of order at least  $\xi + 2$ . If  $Z$  is a subset of  $E(G)$  with  $|Z| \leq \xi$ , then there exists a set of two edges of  $G - Z$  with at least one common end free with respect to  $\mathcal{E} - Z$ .*

**Proof.** Suppose that every set of two edges of  $G - Z$  with at least one common end is not free with respect to  $\mathcal{E} - Z$ . That is, for every edges  $e_1, e_2 \in E(G) - Z$ , there exist  $Y \subset \{e_1, e_2\}$  and  $[A, B] \in \mathcal{E} - (Y \cup Z)$  of order at most  $1 - |Y|$  such that every edge in  $\{e_1, e_2\} - Y$  has both ends in  $A$ . Let  $G_1, \dots, G_c$  be the components of  $G - Z$ . Since  $\mathcal{E} - Z$  has order at least two,

there uniquely exists  $i$  with  $1 \leq i \leq c$  such that  $[V(G) - V(G_i), V(G_i)] \in \mathcal{E} - Z$  by (E1), (E2) and (E3). Without loss of generality, we may assume that  $[V(G) - V(G_1), V(G_1)] \in \mathcal{E} - Z$ . Define  $\mathcal{E}'$  to be the set of edge-cuts of  $G_1$  such that  $[A, B] \in \mathcal{E}'$  if and only if  $[A, B]$  has order less than two and  $[A \cup \bigcup_{i=2}^c V(G_i), B] \in \mathcal{E} - Z$ . It is clear that  $\mathcal{E}'$  is an edge-tangle in  $G_1$  of order two.

It is well-known that there exist a tree  $T$  and a partition  $(X_t : t \in V(T))$  of  $V(G_1)$  such that

- $G_1[X_t]$  is either a vertex or 2-edge-connected for every  $t \in V(T)$ ,
- for every adjacent vertices  $t_1, t_2$  of  $T$ , there exists uniquely one edge between  $X_{t_1}$  and  $X_{t_2}$ , and
- Every edge of  $G_1$  has both ends in  $X_t$  for some  $t \in V(T)$  or has one end in  $X_{t_1}$  and one end in  $X_{t_2}$  for some adjacent vertices  $t_1, t_2$  of  $T$ .

For each edge  $e = t_1 t_2$  of  $T$ , let  $T_{e,t_1}$  and  $T_{e,t_2}$  be the components of  $T - e$  containing  $t_1$  and  $t_2$ , respectively, and define  $Y_{e,t_1} = \bigcup_{t \in V(T_{e,t_1})} X_t$  and  $Y_{e,t_2} = \bigcup_{t \in V(T_{e,t_2})} X_t$ . Since  $\mathcal{E}'$  has order at least two, by (E1) and (E2), exactly one of  $[Y_{e,t_1}, Y_{e,t_2}]$  and  $[Y_{e,t_2}, Y_{e,t_1}] \in \mathcal{E} - Z$ . If the former happens, then we orientate the edge  $e$  from  $t_1$  to  $t_2$ , otherwise, we orientate the edge  $e$  from  $t_2$  to  $t_1$ . So we obtain an orientation of  $E(T)$  and hence  $T$  has a vertex  $t^*$  of out-degree zero.

Given two edges  $e, f$  of  $G - Z$ , by the assumption, there exist  $Y \subset \{e, f\}$  and  $[A, B] \in \mathcal{E} - Z$  of order at most  $1 - |Y|$  such that the both ends of the edges in  $\{e, f\} - Y$  are in  $A$ . Let  $[A', B']$  be the edge-cut  $[A \cap V(G_1), B \cap V(G_1)]$  of  $G_1$  of order at most  $1 - |Y|$ . By (E1) and (E2),  $[A', B'] \in \mathcal{E}'$ . Since  $[A', B']$  has order at most one,  $B'$  contains  $X_{t^*}$ .

We first claim that  $X_{t^*}$  is a single vertex. Suppose  $X_{t^*}$  contains at least two vertices, then  $G_1[X_{t^*}]$  is 2-edge-connected. We choose  $e, f$  to be two edges of  $G_1[X_{t^*}]$  sharing a common end. Since  $G_1[X_{t^*}]$  is 2-edge-connected, either  $A' \cap X_{t^*} = \emptyset$  or  $B' \cap X_{t^*} = \emptyset$ . Since one of  $e, f$  has both ends in  $A'$ ,  $B' \cap X_{t^*} = \emptyset$ , a contradiction. Hence  $X_{t^*}$  contains exactly one vertex  $v$ .

Since  $\mathcal{E}'$  has order at least two,  $v$  is incident with at least two edges of  $G_1$ . Let  $e, f$  be two edges of  $G_1$  incident with  $v$ . Since one of  $e, f$  has both ends in  $A'$ ,  $v \in A'$ , a contradiction. This proves the lemma. ■

### 3 Excluding immersions

Given a simple graph  $H$ , an  $H$ -minor of a graph  $G$  is a map  $\alpha$  with domain  $V(H)$  such that

- $\alpha(h)$  is a nonempty connected subgraph of  $G$ , for every  $h \in V(H)$ ;
- if  $h_1$  and  $h_2$  are different vertices in  $H$ , then  $\alpha(h_1)$  and  $\alpha(h_2)$  are disjoint;
- if  $h_1h_2$  is an edge in  $H$ , then there exists an edge of  $G$  with one end in  $\alpha(h_1)$  and one end in  $\alpha(h_2)$ .

We say that  $G$  contains an  $H$ -minor if such a function  $\alpha$  exists. And for every  $h \in V(H)$ ,  $\alpha(h)$  is called a *branch set* of  $\alpha$ .

Given a simple graph  $H$ , an  $H$ -thorns of a graph  $G$  is a map  $\alpha$  with domain  $V(H)$  such that

- $\alpha(h)$  is a connected subgraph of  $G$  with at least one edge, for every  $h \in V(H)$ ;
- if  $h_1$  and  $h_2$  are different vertices in  $H$ , then  $\alpha(h_1)$  and  $\alpha(h_2)$  are edge-disjoint;
- if  $h_1h_2$  is an edge in  $H$ , then  $V(\alpha(h_1)) \cap V(\alpha(h_2)) \neq \emptyset$ ;

We say that  $G$  contains an  $H$ -thorns if such a function  $\alpha$  exists. And for every  $h \in V(H)$ ,  $\alpha(h)$  is called a *branch set* of  $\alpha$ .

Note that if a graph contains a vertex  $v$  incident with  $d$  edges, then it contains a  $K_d$ -thorns whose branch sets are the edges incident with  $v$ . Another example of thorns is that every  $r \times r$ -grid contains a  $K_r$ -thorns by defining  $\alpha(v_i)$  to be the union of the  $i$ -th row and the  $i$ -th column.

**Lemma 3.1.** *If  $H$  is a simple graph, then a graph  $G$  contains  $H$ -thorns if and only if  $L(G)$  contains an  $H$ -minor.*

**Proof.** Let  $\alpha$  be an  $H$ -thorns in  $G$ . For every  $h \in V(H)$ , define  $\beta(h)$  to be the subgraph of  $L(G)$  induced by  $E(\alpha(h))$ . It is clear that  $\beta$  is an  $H$ -minor in  $L(G)$ .

Let  $\beta'$  be an  $H$ -minor in  $L(G)$ . For every  $h \in V(H)$ , define  $\alpha'(h)$  to be the connected subgraph of  $G$  with  $E(\alpha'(h)) = V(\beta'(h))$ . Then it is obvious that  $\alpha'$  is an  $H$ -thorns in  $G$ . ■

The following was proved by Robertson and Seymour [8].

**Lemma 3.2** ([8]). *Let  $G$  be a simple graph, and let  $Z$  be a subset of  $V(G)$  with  $|Z| = \xi$ . Let  $k \geq \lceil \frac{3}{2}\xi \rceil$ , and let  $\alpha$  be a  $K_k$ -minor in  $G$ . If there is no separation  $(A, B)$  of  $G$  of order less than  $|Z|$  such that  $Z \subseteq V(A)$  and  $A \cap \alpha(h) = \emptyset$  for some  $h \in V(K_k)$ , then for every partition  $(Z_1, \dots, Z_n)$  of  $Z$  into non-empty subsets, there are  $n$  connected subgraphs  $T_1, \dots, T_n$  of  $G$ , mutually disjoint and  $V(T_i) \cap Z = Z_i$  for  $1 \leq i \leq n$ .*

Now, we prove an edge-variant of Lemma 3.2.

**Lemma 3.3.** *Let  $G$  be a graph, and let  $X$  be a subset of  $E(G)$  with size  $\xi$ . Let  $k \geq \lceil \frac{3}{2}\xi \rceil$ , and let  $\alpha$  be a  $K_k$ -thorns in  $G$ . If there is no  $Y \subseteq X$  and edge-cut  $[A, B]$  of  $G - Y$  order less than  $\xi - |Y|$  such that every edge in  $X - Y$  is incident with some vertices in  $A$  and  $A \cap V(\alpha(h)) = \emptyset$  for some  $h \in V(K_k)$ , then for every partition  $(X_1, \dots, X_n)$  of  $X$  into non-empty subsets, there are  $n$  connected subgraphs  $T_1, \dots, T_n$  of  $G$ , mutually edge-disjoint and  $E(T_i) \cap X = X_i$  for  $1 \leq i \leq n$ .*

**Proof.** Let  $\beta$  be the  $K_k$ -minor in  $L(G)$  corresponding to  $\alpha$  mentioned in Lemma 3.1, and let  $X'$  be the set of vertices in  $L(G)$  corresponding to  $X$ . Suppose that there exists a separation  $(A', B')$  of  $L(G)$  of order less than  $\xi$  such that  $X' \subseteq V(A')$  and  $A' \cap \beta(h) = \emptyset$  for some  $h \in V(K_k)$ . We may assume that the order of  $(A', B')$  is as small as possible. So every vertex in  $V(A') \cap V(B') - X'$  must have a neighbor in  $V(A') - V(B')$  and a neighbor in  $V(B') - V(A')$ , and every vertex in  $V(A') \cap V(B') \cap X'$  has a neighbor in  $V(B') - V(A')$ . Define  $B = \{v \in V(G) : V(\text{cl}(v)) \subseteq V(B')\}$  and  $A = V(G) - B$ . Then  $[A, B]$  is an edge-cut of  $G$ . Let  $Y$  be the subset of  $X$  consisting of the edges in  $X$  with both ends in  $B$ . Note that the order of  $[A, B]$  equals  $|V(A') \cap V(B')| - |\{v \in V(A') \cap V(B') : v \text{ has no neighbor in } V(A') - V(B')\}| = |V(A') \cap V(B')| - |Y| < \xi - |Y|$ . Furthermore, every edge in  $X - Y$  has an end in  $A$ . In addition, every vertex of  $\beta(h)$  is in  $V(B') - V(A')$ , so every edge of  $\alpha(h)$  has both ends in  $B$ . That is,  $V(\alpha(h)) \cap A = \emptyset$ , a contradiction. Therefore, there does not exist a separation  $(A', B')$  of  $G$  of order less than  $\xi$  such that  $X' \subseteq V(A')$  and  $A' \cap \beta(h) = \emptyset$  for some  $h \in V(K_k)$ .

Let  $(X_1, X_2, \dots, X_n)$  be a partition of  $X$  into nonempty sets. Let  $(Z_1, Z_2, \dots, Z_n)$  be the partition of  $X'$  such that  $Z_i$  is the set of the corresponding edges in  $X_i$  for  $1 \leq i \leq n$ . By Lemma 3.2, there exist mutually disjoint connected subgraphs  $T'_1, \dots, T'_n$  of  $L(G)$  such that  $V(T'_i) \cap X' = Z_i$  for every  $1 \leq i \leq n$ . For every  $1 \leq i \leq n$ , define  $T_i$  to be the connected subgraph



of  $G$  with  $E(T_i) = V(T'_i)$ . Then  $T_1, \dots, T_n$  are mutually edge-disjoint and  $E(T_i) \cap X = X_i$ . ■

A tangle  $\mathcal{T}$  in  $G$  controls an  $H$ -minor  $\alpha$  if there do not exist  $(A, B) \in \mathcal{T}$  of order less than  $|V(H)|$  and  $h \in V(H)$  such that  $V(\alpha(h)) \subseteq V(A)$ . An edge-tangle  $\mathcal{E}$  in  $G$  controls an  $H$ -thorns  $\alpha$  if  $V(\alpha(h)) \cap B \neq \emptyset$  for every  $h \in V(H)$  and  $[A, B] \in \mathcal{E}$  of order less than  $|V(H)|$ .

The *degree sequence* of a graph  $G$  is the decreasing sequence of the degrees of the vertices of  $G$ .

**Lemma 3.4.** *Let  $G$  be a graph and  $H$  be a graph on  $h$  vertices with degree sequence  $(d_1, d_2, \dots, d_h)$ . Let  $d = d_1$  and  $t \geq \lceil \frac{3hd}{2} \rceil$ . Let  $\mathcal{E}$  be an edge-tangle of order at least  $hd$  in  $G$  that controls a  $K_t$ -thorns. Assume that there exist pairwise disjoint subsets  $X_1, X_2, \dots, X_h$  of  $E(G)$  such that  $\bigcup_{i=1}^h X_i$  is free with respect to  $\mathcal{E}$ , and for each  $i$  with  $1 \leq i \leq h$ ,  $X_i$  consists of  $d_i$  edges incident with a common vertex  $v_i$ . If  $v_1, v_2, \dots, v_h$  are distinct, then  $G$  has an  $H$ -immersion  $(\pi_V, \pi_E)$  with  $\pi_V(V(H)) = \{v_1, v_2, \dots, v_h\}$ .*

**Proof.** Let  $\alpha$  be a  $K_t$ -thorns in  $G$  controlled by  $\mathcal{E}$ , and let  $X = \bigcup_{i=1}^h X_i$ . Suppose that there exists  $Y \subseteq X$  and an edge-cut  $[A, B]$  of  $G - Y$  of order less than  $|X - Y|$  such that every edge in  $X - Y$  is incident with vertices in  $A$  and  $A \cap V(\alpha(u)) = \emptyset$  for some  $u \in V(K_t)$ . We assume that  $Y$  is maximal, so every edge in  $X - Y$  has both ends in  $A$ . Since  $X$  is free with respect to  $\mathcal{E}$ ,  $[A, B] \notin \mathcal{E} - Y$ . But the order of  $[A, B]$  is at less than  $|X - Y| = hd - |Y|$ . So  $[B, A] \in \mathcal{E} - Y$  by (E1). However,  $\mathcal{E}$  controls  $\alpha$ , so  $A \cap V(\alpha(u)) \neq \emptyset$ , a contradiction. Therefore, by Lemma 3.3, for every partition  $(Z_1, Z_2, \dots, Z_{|E(H)|})$  of  $X$  there exist pairwise edge-disjoint connected subgraphs  $T_1, T_2, \dots, T_{|E(H)|}$  of  $G$  such that  $E(T_i) \cap X = Z_i$  for every  $1 \leq i \leq |E(H)|$ .

Let  $V(H) = \{u_1, u_2, \dots, u_h\}$ , where  $\deg_H(u_i) = d_i$  for every  $1 \leq i \leq h$ , and let  $E(H) = \{e_1, e_2, \dots, e_{|E(H)|}\}$ . For every  $i$  with  $1 \leq i \leq h$ , we pick  $Y_i \subseteq X_i$  with  $|Y_i| = \deg_H(u_i)$ . For every  $1 \leq i \leq h$ , define an onto function  $f_i$  from  $Y_i$  to the set of edges of  $H$  incident with  $u_i$  such that the preimage of every non-loop edge incident with  $u_i$  has size one and the preimage of every loop incident with  $u_i$  has size two. So for every edge  $e_i$  of  $H$ , there exist exactly two edges in  $\bigcup_{i=1}^h Y_i$  mapped to  $e_i$  by the functions  $f_1, f_2, \dots, f_h$ , and we denote the set of these two edges as  $Z_i$ . So  $(Z_1, Z_2, \dots, Z_{|E(H)|})$  is a partition of  $\bigcup_{i=1}^h Y_i$  into non-empty sets. Then the union of the mentioned pairwise

edge-disjoint trees  $T_1, \dots, T_{|E(H)|}$  in  $G$  defines an  $H$ -immersion  $(\pi_V, \pi_E)$  in  $G$  with  $\pi_V(V(H)) = \{v_1, v_2, \dots, v_h\}$ . ■

The following was proved in [4].

**Lemma 3.5.** [4] *Let  $G$  be a graph and  $\mathcal{T}$  a tangle in  $G$  of order  $\theta$ , and let  $c$  be a positive integer. For every  $1 \leq i \leq c$ , let  $d_i, k_i$  be positive integers, and let  $\{X_{i,j} \subseteq V(G) : j \in J_i\}$  be a family of subsets of  $V(G)$  indexed by  $J_i$ . Let  $d, k$  be integers such that  $\theta \geq (kcd)^{d+1} + d$ ,  $d_i \leq d$  and  $k_i \leq k$  for  $1 \leq i \leq c$ . Let  $J_i^* \subseteq J_i$  with  $|J_i^*| \leq k_i$  for each  $1 \leq i \leq c$ , such that  $\bigcup_{i=1}^c \bigcup_{j \in J_i^*} X_{i,j}$  is free with respect to  $\mathcal{T}$  and  $X_{i,j} \cap X_{i',j'} = \emptyset$  for distinct pairs  $(i, j), (i', j')$  with  $1 \leq i \leq i' \leq c$ ,  $j \in J_i^*$  and  $j' \in J_{i'}^*$ . If  $|X_{i,j}| \leq d_i$  for every  $1 \leq i \leq c$  and  $j \in J_i$ , then either*

1. *there exist  $J'_1, J'_2, \dots, J'_c$  with  $J_i^* \subseteq J'_i \subseteq J_i$  and  $|J'_i| = k_i$  for each  $1 \leq i \leq c$  such that  $\bigcup_{1 \leq i \leq c} \bigcup_{j \in J'_i} X_{i,j}$  is free with respect to  $\mathcal{T}$ , and  $X_{i,j} \cap X_{i',j'} = \emptyset$  for all distinct pairs  $(i, j), (i', j')$  with  $1 \leq i \leq i' \leq c'$  and  $j \in J'_i, j' \in J'_{i'}$ , or*
2. *there exist  $Z \subseteq V(G)$  with  $|Z| \leq (kcd)^{d+1}$  and integer  $i$  with  $1 \leq i \leq c$  and  $|J_i^*| < k_i$  such that for every  $j \in J_i$ , either  $X_{i,j} \cap Z \neq \emptyset$ , or  $X_{i,j}$  is not free with respect to  $\mathcal{T} - Z$ .*

**Lemma 3.6.** *Let  $G$  be a graph and  $\mathcal{E}$  an edge-tangle in  $G$  of order  $\theta$ , and let  $c$  be a positive integer. For every  $1 \leq i \leq c$ , let  $d_i, k_i$  be positive integers, and let  $\{X_{i,j} \subseteq E(G) : j \in J_i\}$  be a family of subsets of  $E(G)$  indexed by  $J_i$ . Let  $d, k$  be integers such that  $\theta \geq 3(kcd)^{d+1} + 3d$ ,  $d_i \leq d$  and  $k_i \leq k$  for  $1 \leq i \leq c$ . Let  $J_i^* \subseteq J_i$  with  $|J_i^*| \leq k_i$  for each  $1 \leq i \leq c$ , such that  $\bigcup_{i=1}^c \bigcup_{j \in J_i^*} X_{i,j}$  is free with respect to  $\mathcal{E}$  and  $X_{i,j} \cap X_{i',j'} = \emptyset$  for distinct pairs  $(i, j), (i', j')$  with  $1 \leq i \leq i' \leq c$ ,  $j \in J_i^*$  and  $j' \in J_{i'}^*$ . If  $|\bigcup_{i=1}^c \bigcup_{j \in J_i^*} X_{i,j}| \leq \theta/3$  and  $|X_{i,j}| \leq d_i$  for every  $1 \leq i \leq c$  and  $j \in J_i$ , then either*

1. *there exist  $J'_1, J'_2, \dots, J'_c$  with  $J_i^* \subseteq J'_i \subseteq J_i$  and  $|J'_i| = k_i$  for each  $1 \leq i \leq c$  such that  $\bigcup_{1 \leq i \leq c} \bigcup_{j \in J'_i} X_{i,j}$  is free with respect to  $\mathcal{E}$ , and  $X_{i,j}$  and  $X_{i',j'}$  are disjoint for all distinct pairs  $(i, j), (i', j')$  with  $1 \leq i \leq i' \leq c'$  and  $j \in J'_i, j' \in J'_{i'}$ , or*
2. *there exist  $Z \subseteq E(G)$  with  $|Z| \leq (kcd)^{d+1}$  and integer  $i$  with  $1 \leq i \leq c$  and  $|J_i^*| < k_i$  such that for every  $j \in J_i$ , either  $X_{i,j} \cap Z \neq \emptyset$ , or  $X_{i,j}$  is not free with respect to  $\mathcal{E} - Z$ .*

**Proof.** Since  $\mathcal{E}$  is an edge-tangle of order  $\theta$  in  $G$ ,  $\overline{\mathcal{E}}$  is an tangle of order  $\theta/3 \geq (kcd)^{d+1} + d$  in  $L(G)$  by Lemma 2.3. For every  $1 \leq i \leq c$  and  $j \in J_i$ , define  $Y_{i,j}$  to be the subset of  $V(L(G))$  corresponding to the edges in  $X_{i,j}$ . Since  $|\bigcup_{i=1}^c \bigcup_{j \in J_i^*} X_{i,j}| \leq \theta/3$ ,  $\bigcup_{i=1}^c \bigcup_{j \in J_i^*} Y_{i,j}$  is free with respect to  $\overline{\mathcal{E}}$  by Lemma 2.5. So by Lemma 3.5, either

- there exist  $J'_1, J'_2, \dots, J'_c$  with  $J_i^* \subseteq J'_i \subseteq J_i$  and  $|J'_i| = k_i$  for each  $1 \leq i \leq c$  such that  $\bigcup_{1 \leq i \leq c} \bigcup_{j \in J'_i} Y_{i,j}$  is free with respect to  $\overline{\mathcal{E}}$ , and  $Y_{i,j}$  and  $Y_{i',j'}$  are disjoint for all distinct pairs  $(i,j), (i',j')$  with  $1 \leq i \leq i' \leq c'$  and  $j \in J'_i, j' \in J'_{i'}$ , or
- there exist  $Z' \subseteq V(L(G))$  and integer  $i^*$  with  $|Z'| \leq (kcd)^{d+1}$  and  $1 \leq i^* \leq c$  satisfying that for every  $j \in J_{i^*}$ , either  $Y_{i^*,j} \cap Z' \neq \emptyset$ , or  $Y_{i^*,j}$  is not free with respect to  $\overline{\mathcal{E}} - Z'$ .

Note that  $X_{i,j}$  and  $X_{i',j'}$  are disjoint for all distinct pairs  $(i,j), (i',j')$  with  $1 \leq i \leq i' \leq c$ ,  $j \in J'_i$  and  $j' \in J'_{i'}$ . If the first statement above holds, then  $\bigcup_{i=1}^c \bigcup_{j \in J'_i} X_{i,j}$  is free with respect to  $\mathcal{E}$  by Lemma 2.6, and hence the first statement of the lemma holds. So we may assume that such  $Z'$  and  $i^*$  mentioned in the above second statement exists.

We shall prove the second statement of the lemma holds. Let  $Z$  be the set of edges of  $G$  corresponding to  $Z'$ , and let  $j$  be an arbitrary element of  $J_{i^*}$ . If  $Y_{i^*,j} \cap Z' \neq \emptyset$ , then  $X_{i^*,j} \cap Z \neq \emptyset$ . So we may assume that  $Y_{i^*,j}$  is not free with respect to  $\overline{\mathcal{E}} - Z'$ . Therefore,  $Y_{i^*,j} \cup Z'$  is not free with respect to  $\overline{\mathcal{E}}$ .

Suppose that  $X_{i^*,j}$  is free with respect to  $\mathcal{E} - Z$ . So  $X_{i^*,j} \cup Z$  is free with respect to  $\mathcal{E}$ . By Lemma 2.5,  $Y_{i^*,j} \cup Z'$  is free with respect to  $\overline{\mathcal{E}}$ , a contradiction. Consequently, the second statement of the lemma holds. ■

A family  $\mathcal{D}$  of edge-cuts of a graph is *cross-free* if  $A \cap C = \emptyset$  for every pair of edge-cuts  $[A, B], [C, D]$  in  $\mathcal{D}$ . The following is a structure theorem for excluding a fixed graph as an immersion in a graph with an edge-tangle controlling a big complete graph-thorns.

**Theorem 3.7.** *For every positive integers  $d, h$ , there exist positive integers  $\theta, \xi$  such that the following holds. If  $H$  is a graph with degree sequence  $(d_1, d_2, \dots, d_h)$ , where  $d_1 = d$ , and  $G$  is a graph that does not contain an  $H$ -immersion, then for every edge-tangle  $\mathcal{E}$  of order at least  $\theta$  in  $G$  controlling a  $K_{\lceil \frac{3}{2}dh \rceil}$ -thorns, there exist  $C \subseteq E(G)$  with  $|C| \leq \xi$ ,  $U \subseteq V(G)$  with*

$|U| \leq h - 1$  and a cross-free family  $\mathcal{D} \subseteq \mathcal{E} - C$  such that for every vertex  $v \in V(G) - U$ , there exists  $[A, B] \in \mathcal{D}$  of order at most  $d_{|U|+1} - 1$  with  $v \in A$ .

**Proof.** Define  $\theta = 3(hd)^{d+1} + 3d$  and  $\xi = (hd)^{d+1}$ . Let  $H$  be a graph on  $h$  vertices with degree sequence  $(d_1, d_2, \dots, d_h)$  and  $d_1 = d$  and  $G$  a graph with no  $H$ -immersion, and let  $\mathcal{E}$  be an edge-tangle in  $G$  of order at least  $\theta$  controlling a  $K_{\lceil \frac{3}{2}dh \rceil}$ -thorns.

Note that it is sufficient to prove this theorem for the case that  $H$  has no isolated vertices. Since every graph with at least  $h$  vertices containing an  $H'$ -immersion contains an  $H$ -immersion, where  $H'$  is the graph obtained from  $H$  by deleting all isolated vertices. So we may assume that  $H$  has no isolated vertices.

We first prove that there exist  $C \subseteq E(G)$  with  $|C| \leq \xi$  and  $U \subseteq V(G)$  with  $|U| \leq h - 1$  such that for every  $v \in V(G) - U$ , there exists  $[A_v, B_v] \in \mathcal{E} - C$  of order at most  $d_{|U|+1} - 1$  such that  $v \in A$ .

Define  $\mathcal{S}_0^* = \emptyset$  and  $U_0 = \emptyset$ . For each  $i$  with  $1 \leq i \leq d$ , define  $U_i$  to be a subset of  $V(G)$  and define  $\mathcal{S}_i^*$  to be a maximal collection of pairwise disjoint sets of  $d - i + 1$  edges of  $G$  with the following properties.

- For every  $S \in \mathcal{S}_i^*$ , the edges in  $S$  have a common end  $v_S \notin \bigcup_{j=1}^{i-1} U_j$ , and  $S$  is disjoint from  $S'$  for every  $S' \in \bigcup_{j=1}^{i-1} \mathcal{S}_j^*$ .
- For every pair of distinct sets  $S, S' \in \mathcal{S}_i^*$ ,  $v_S \neq v_{S'}$ .
- $\bigcup_{j=1}^i \bigcup_{S \in \mathcal{S}_j^*} S$  is free with respect to  $\mathcal{E}$ .
- $U_i = \{v_S : S \in \mathcal{S}_i^*\}$ .

Define  $r$  to be the smallest integer such that  $|\bigcup_{i=1}^r U_i| < |\{u \in V(H) : \deg_H(u) \geq d - r + 1\}|$ . Note that the number  $r$  exists and  $|\bigcup_{i=1}^r U_i| < h$ , otherwise  $G$  contains an  $H$ -immersion by Lemma 3.4. Let  $U = \bigcup_{i=1}^r U_i$  and  $\mathcal{S}^* = \bigcup_{i=1}^r \mathcal{S}_i^*$ .

For every  $v \in V(G)$ , define  $\mathcal{S}_v$  to be the collection of the sets of  $d_{|U|+1}$  edges incident with  $v$ . Note that  $\mathcal{S}_v = \emptyset$  if  $v$  is incident with less than  $d_{|U|+1}$  edges. Let  $\mathcal{S} = \mathcal{S}^* \cup \bigcup_{v \in V(G)-U} \mathcal{S}_v$ . By the maximality of  $\mathcal{S}^*$ , for every  $X \in \bigcup_{v \in V(G)-U} \mathcal{S}_v$  with  $X \cap (\bigcup_{S \in \mathcal{S}^*} S) = \emptyset$ ,  $X \cup \bigcup_{S \in \mathcal{S}^*} S$  is not free with respect to  $\mathcal{E}$ . So  $\mathcal{S}^*$  is the only set  $\mathcal{S}'$  with the properties that  $\mathcal{S}^* \subseteq \mathcal{S}' \subseteq \mathcal{S}$  and the members of  $\mathcal{S}'$  are pairwise disjoint and  $\bigcup_{X \in \mathcal{S}'} X$  is free. Since  $|\mathcal{S}^*| = |U| < h$ , by Lemma 3.6, there exists  $C \subseteq E(G)$  with  $|C| \leq \xi$  such

that every member of  $\mathcal{S}$  either intersects  $C$  or is not free with respect to  $\mathcal{E} - C$ .

For every  $v \in V(G)$  incident with at most  $d_{|U|+1} - 1$  edges in  $G - C$ ,  $[\{v\}, V(G) - \{v\}]$  is an edge-cut in  $\mathcal{E} - C$  of order at most  $d_{|U|+1} - 1$  such that  $v$  is in the first entry of the edge-cut. Let  $v \in V(G) - U$  incident with at least  $d_{|U|+1}$  edges in  $G - C$ . Then there exists  $X \in \mathcal{S}_v$  such that every edge in  $X$  is incident with  $v$  and  $X \cap C = \emptyset$ . So  $X$  is not free with respect to  $\mathcal{E} - C$ . Hence there exist  $Y \subseteq X$  and an edge-cut  $[A, B] \in \mathcal{E} - (C \cup Y)$  of  $G - (C \cup Y)$  of order less than  $|X - Y|$  such that every edge in  $X - Y$  has both ends in  $A$ . Since every edge in  $X - Y$  is incident with  $v$ , we have that  $v \in A$  and  $[A, B] \in \mathcal{E} - C$  is an edge-cut of  $G - C$  of order less than  $|X| = d_{|U|+1}$ . This proves that for every vertex  $v \in V(G) - U$ , there exists  $[A_v, B_v] \in \mathcal{E} - C$  of order at most  $d_{|U|+1} - 1$  such that  $v \in A_v$ .

Therefore, there exists a family  $\mathcal{D} \subseteq \mathcal{E} - C$  of edge-cuts of  $G - C$  of order at most  $d_{|U|+1} - 1$  such that for every  $v \in V(G) - U$ , there exists  $[A, B] \in \mathcal{D}$  such that  $v \in A$ . We assume that  $\sum_{[A, B] \in \mathcal{D}} |A|$  is as small as possible. We shall show that  $\mathcal{D}$  is cross-free.

Suppose that  $\mathcal{D}$  is not cross-free, then there exist  $[A_1, B_1], [A_2, B_2] \in \mathcal{D}$  such that  $A_1 \cap A_2 \neq \emptyset$ . By the submodularity,  $|[A_1 \cap B_2, B_1 \cup A_2]| + |[A_1 \cup B_2, B_1 \cap A_2]| \leq |[A_1, B_1]| + |[B_2, A_2]| \leq 2(d_{|U|+1} - 1)$ , so one of  $[A_1 \cap B_2, B_1 \cup A_2]$  and  $[B_1 \cap A_2, A_1 \cup B_2]$  has order no more than  $d_{|U|+1} - 1$ . By symmetry, we may assume that  $[A_1 \cap B_2, B_1 \cup A_2]$  has order no more than  $d_{|U|+1} - 1$ . By (E1) and (E2),  $[A_1 \cap B_2, B_1 \cup A_2] \in \mathcal{E} - C$ . Let  $\mathcal{D}' = (\mathcal{D} - \{[A_1, B_1]\}) \cup \{[A_1 \cap B_2, B_1 \cup A_2]\}$ . Since  $A_1 \subseteq (A_1 \cap B_2) \cup A_2$ ,  $\mathcal{D}'$  is contained in  $\mathcal{E} - C$  and is a family of edge-cuts of  $G - C$  of order at most  $d_{|U|+1} - 1$  such that for every  $v \in V(G) - U$ , there exists  $[A, B] \in \mathcal{D}'$  such that  $v \in A$ . Hence, by the minimality of  $\mathcal{D}$ ,  $|A_1 \cap B_2| \geq |A_1|$ . This implies that  $A_1 = A_1 \cap B_2 \subseteq B_2$ , so  $A_1 \cap A_2 = \emptyset$ , a contradiction. Therefore,  $\mathcal{D}$  is cross-free. ■

Let  $G$  be a graph and  $S$  a subgraph of  $G$ . We define  $S_G^+$  to be the graph obtained from  $S$  by attaching  $\deg_G(v) - \deg_S(v)$  leaves to  $v$ , for each  $v \in V(S)$ , where  $\deg_G(v)$  and  $\deg_S(v)$  are the degree of  $v$  in  $G$  and  $S$ , respectively. So every vertex in  $V(S_G^+) - V(S)$  corresponds to an edge in  $E(G) - E(S)$ . Note that if  $e$  is an edge in  $E(G) - E(S)$  with both ends in  $V(S)$ , then  $e$  contributes two leaves to  $S_G^+$ , where one is adjacent to  $u$  and one is adjacent to  $v$ . In particular, if  $e \in E(G) - E(S)$  is a loop incident with a vertex  $v$  in  $S$ , then  $e$  contributes two leaves adjacent to  $v$  in  $S_G^+$ . When the supergraph  $G$  is clear, we denote  $S_G^+$  by  $S^+$  for short.

Let  $G$  and  $H$  be graphs, and let  $S, R$  be subgraphs of  $G, H$ , respectively. We say that  $S_G^+$  realizes  $R_H^+$  if  $S_G^+$  contains a  $R_H^+$ -immersion  $(\pi_V, \pi_E)$  such that  $\pi_V(V(R_H^+) - V(R)) \subseteq V(S_G^+) - V(S)$  and  $\pi_V(V(R)) \subseteq V(S)$ .

Let  $H$  be a graph. A *shell* of  $H$  is a collection of disjoint connected subgraphs of  $H$  such that every vertex of  $H$  is contained in a member of  $H$ . The following is the essential step toward the main lemma of this section.

**Lemma 3.8.** *For every connected graph  $H$  and for every positive integer  $k$ , there exist integers  $\theta, w, \xi$  such that the following holds. If  $G$  is a graph that does not contain  $k$  edge-disjoint  $H$ -immersions and  $\mathcal{E}$  is an edge-tangle of  $G$  of order at least  $\theta$  controlling a  $K_w$ -thorns, then there exist  $Z \subseteq E(G)$  with  $|Z| \leq \xi$  and  $[A, B] \in \mathcal{E} - Z$  of order zero such that  $G[A]$  contains all  $H$ -immersions in  $G - Z$ .*

**Proof.** Let  $H$  be a graph with degree sequence  $(d_1, d_2, \dots, d_h)$ , where  $h = |V(H)|$ . Let  $k$  be a fixed integer in this proof. Let  $\theta_0$  be the number  $\theta$  mentioned in Theorem 3.7 by taking  $h = h$  and  $d = kd_1$ . Define  $\xi = (6k^3h^4d_1^2)^{3kd_1h}$ ,  $\theta = \theta_0 + 4\xi$  and  $w = \frac{3}{2}(khd_1)^2 + \xi$ . Let  $G$  be a graph and  $\mathcal{E}$  an edge-tangle of order  $\theta$  in  $G$  controlling a  $K_w$ -thorns  $\alpha$ .

Define  $H_k$  to be the graph obtained from  $H$  by duplicating each edge  $k$  times. Note that  $H_k$  is a graph on  $h$  vertices with maximum degree  $kd_1$ . If  $G$  contains an  $H_k$ -immersion, then  $G$  contains  $k$  edge-disjoint  $H$ -immersions. Let  $\xi_0 = (hkd_1)^{kd_1+1} + 2(h-1)^2kd_1$ . Since  $G$  does not contain  $k$  edge-disjoint  $H$ -immersions,  $G$  does not contain an  $H_k$ -immersion. By Theorem 3.7, there exist  $Z'_0 \subseteq E(G)$  with  $|Z'_0| \leq (hkd_1)^{kd_1+1}$ ,  $U \subseteq V(G)$  with  $|U| \leq h-1$  and  $\mathcal{F} \subseteq \mathcal{E} - Z'_0$  is a cross-free family of edge-cuts  $[A_v, B_v]$  of order less than  $kd_1$  with  $v \in A_v$  for each  $v \in V(G) - U$ .

Suppose that we cannot find for every  $u \in U$ , there exists a set  $S_u$  of  $kd_1$  edges incident with  $u$  such that  $S_{u'} \cap S_{u''} = \emptyset$  for distinct  $u, u' \in U$ . Then by Hall's condition, we can further delete at most  $(h-1)kd_1$  edges from  $G - Z'_0$  and put those edges into  $Z'_0$  to reduce the number of vertices in  $U$ . So we assume that for every  $u \in U$ , there exists a set of  $kd_1$  edges  $S_u$  incident with  $u$  such that  $S_{u'} \cap S_{u''} = \emptyset$  for distinct  $u, u' \in U$ . If  $\bigcup_{u \in U} S_u$  is not free with respect to  $\mathcal{E} - Z'_0$ , then there exists  $Y \subseteq \bigcup_{u \in U} S_u$  and  $[A, B] \in \mathcal{E} - Z'_0$  of order less than  $|\bigcup_{u \in U} S_u - Y|$  such that every edge in  $\bigcup_{u \in U} S_u - (Z'_0 \cup Y)$  has both ends in  $A$ . This implies that some vertex in  $U$  belongs to  $A$ . So we can further delete at most  $(h-1)kd_1$  edges to reduce the number of vertices in  $U$ . In other words, there exists a superset  $Z_0$  of  $Z'_0$  with  $|Z_0| \leq \xi_0$  such that for every  $u \in U$ , there exists a set  $S_u$  of  $kd_1$  edges incident with  $u$  such

that  $S_{u'} \cap S_{u''} = \emptyset$  for distinct  $u, u' \in U$ , and  $\bigcup_{u \in U} S_u$  is free with respect to  $\mathcal{E} - Z_0$ .

We are done if  $G - Z_0$  contains no  $H$ -immersion, so we may assume that  $G - Z_0$  contains an  $H$ -immersion. Since  $|U| \leq h - 1$ , for every  $H$ -immersion  $L = (\pi_V, \pi_E)$  in  $G - Z_0$ , there exists  $v \in V(H)$  such that  $\pi_V(v) \notin U$ . For each  $H$ -immersion  $L = (\pi_V, \pi_E)$  in  $G$ , define  $[A_L, B_L]$  to be the edge-cut of  $G - Z_0$  such that  $A_L = \bigcup_{v \in V(H), \pi_V(v) \notin U} A_{\pi_V(v)}$ . Hence the branch vertices of each  $L$  are contained in  $U \cup A_L$ . Note that the order of each  $[A_L, B_L]$  is at most  $kd_1h$ , so  $[A_L, B_L] \in \mathcal{E} - Z_0$ . Furthermore,  $G[A_L]_G^+$  realizes  $S_H^+$  for some member  $S$  in some shell of  $H$ . In addition, since  $H$  is connected, by deleting components of  $G[A_L] - Z_0$  disconnected to all edges between  $A_L$  and  $B_L$ , we may assume that for every vertex  $A_L$ , there exists a path in  $G[A_L]$  from this vertex to an end of an edge between  $A_L$  and  $B_L$ , unless  $[A_L, B_L]$  has order zero in  $G - Z_0$ .

For every shell  $\mathcal{P}$  of  $H$ , define  $D_{\mathcal{P}}$  to be a subset of  $\{v \in V(H) : \{v\} \in \mathcal{P}\}$  with size at most  $|U|$  and define  $H_{\mathcal{P}}$  to be the graph obtained from the disjoint union of  $k$  copies of  $H$  by identifying each vertex in  $D_{\mathcal{P}}$  in each copy into a vertex. Note that  $G$  contains no  $H_{\mathcal{P}}$ -immersion, otherwise  $G$  contains  $k$  edge-disjoint  $H$ -immersions. Define  $\mathcal{Q}_{\mathcal{P}}$  to be the shell of  $H_{\mathcal{P}}$  consisting of  $\{v\}$ , for each  $v \in D_{\mathcal{P}}$ , and the members of  $\mathcal{P} - \{\{v\} : v \in D_{\mathcal{P}}\}$  in each copy of  $H$ . Define  $\mathcal{X}_0 = \{S_u : u \in U\}$ . Recall that  $\bigcup_{X \in \mathcal{X}_0} X$  is free with respect to  $\mathcal{E} - Z_0$ . For each  $S \in \mathcal{Q}_{\mathcal{P}}$ , define  $\mathcal{X}_S$  to be the collection of all sets of edges satisfying that every set in  $\mathcal{X}_S$  consists of the edges between  $A_L$  and  $B_L$ , for some  $H$ -immersion  $L$  in  $G - Z_0$  such that  $G[A_L]_G^+$  realizes  $S_H^+$ . Define  $\mathcal{X}_E$  to be the collection of the 2-element subsets of  $E(G - Z_0)$  each consisting of two edges having at least one common end. Define  $\mathcal{X}_0^* = \mathcal{X}_0$ ,  $k_0 = |U|$ ,  $\mathcal{X}_E^* = \emptyset$ ,  $k_E = khd_1$ ,  $\mathcal{X}_S^* = \emptyset$  and  $k_S = kh$  for each  $S \in \mathcal{Q}_{\mathcal{P}} - \{\{v\} : v \in D_{\mathcal{P}}\}$ . Note that  $|\mathcal{Q}_{\mathcal{P}}| \leq kh$ .

By Lemma 3.6, for each shell  $\mathcal{P}$  of  $H$ , one of the following holds.

- There exist a collection  $\mathcal{X}'_0$  of size  $k_0$  with  $\mathcal{X}_0^* \subseteq \mathcal{X}'_0 \subseteq \mathcal{X}_0$ , a collection  $\mathcal{X}'_E$  of size  $khd_1$  with  $\mathcal{X}_E^* \subseteq \mathcal{X}'_E \subseteq \mathcal{X}_E$  and collections  $\mathcal{X}'_S$  of size  $k_S$  with  $\mathcal{X}_S^* \subseteq \mathcal{X}'_S \subseteq \mathcal{X}_S$  for each  $S \in \mathcal{Q}_{\mathcal{P}} - \{\{v\} : v \in D_{\mathcal{P}}\}$  such that  $\mathcal{X}'_0 \cup \mathcal{X}'_E \cup \bigcup_{S \in \mathcal{Q}_{\mathcal{P}} - \{\{v\} : v \in D_{\mathcal{P}}\}} \mathcal{X}'_S$  consists of pairwise disjoint members, and the union of its members is free with respect to  $\mathcal{E} - Z_0$ .
- There exist  $Z_{\mathcal{P}} \subseteq E(G) - Z_0$  with  $|Z_{\mathcal{P}}| \leq (k^2(|\mathcal{Q}_{\mathcal{P}}| - |D_{\mathcal{P}}| + 2)h^2d_1^2)^{kd_1h+1}$  and  $S \in \mathcal{Q}_{\mathcal{P}}$  such that for every  $X \in \mathcal{X}_S$ , either  $X \cap Z_{\mathcal{P}} \neq \emptyset$ , or  $X$  is not free with respect to  $\mathcal{E} - (Z_0 \cup Z_{\mathcal{P}})$ .

- There exists  $Z_{\mathcal{P}} \subseteq E(G) - Z_0$  with  $|Z_{\mathcal{P}}| \leq (k^2(|\mathcal{Q}_{\mathcal{P}}| - |D_{\mathcal{P}}| + 2)h^2d_1^2)^{kd_1h+1}$  such that every set of two edges of  $G - (Z_0 \cup Z_{\mathcal{P}})$  sharing a common end is not free with respect to  $\mathcal{E} - (Z_0 \cup Z_{\mathcal{P}})$ .

Notice that  $|\mathcal{X}_0^*| = k_0$ . In addition, the third statement cannot hold for all shells  $\mathcal{P}$  by Lemma 2.11.

Suppose that the first statement holds for some shell  $\mathcal{P}$ . We shall derive a contradiction by showing that  $G$  contains  $k$  edge-disjoint  $H$ -immersions. Let  $X$  be the union of the edges in  $\mathcal{X}'_0 \cup \bigcup_{S \in \mathcal{Q}_{\mathcal{P}} - \{\{v\}: v \in D_{\mathcal{P}}\}} \mathcal{X}'_S$ , so  $X$  is free with respect to  $\mathcal{E} - Z_0$ . Since  $\alpha$  is  $K_w$ -thorns controlled by  $\mathcal{E}$ , there exists a  $K_{w-\xi_0}$ -thorns  $\alpha'$  in  $G - Z_0$  controlled by  $\mathcal{E} - Z_0$ . Suppose that there exist  $Y \subseteq X$  and an edge-cut  $[A, B]$  of  $G - (Z_0 \cup Y)$  of order less than  $|X| - |Y|$  such that every edge in  $X - Y$  is incident with vertices in  $A$  and  $A \cap V(\alpha'(t)) = \emptyset$  for some  $t \in V(K_{w-\xi_0})$ . By (E1),  $[A, B]$  or  $[B, A]$  is in  $\mathcal{E} - (Z_0 \cup Y)$ , and hence  $[A, B]$  or  $[B, A]$  is in  $\mathcal{E} - Z_0$ . Since  $X$  is free with respect to  $\mathcal{E} - Z_0$ ,  $[A, B] \notin \mathcal{E} - Z_0$ . Since  $\mathcal{E} - Z_0$  controls  $\alpha'$ ,  $[B, A] \notin \mathcal{E} - Z_0$ , a contradiction. So no such  $Y$  and  $[A, B]$  exist.

For each  $S \in \mathcal{Q}_{\mathcal{P}}$  and  $X_S \in \mathcal{X}_S$ , define  $L_S$  to be an  $H$ -immersion  $(\pi_V^{(S)}, \pi_E^{(S)})$  in  $G - Z_0$  such that  $G[A_{L_S}]_G^+$  realizes  $S_H^+$ , where  $[A_{L_S}, B_{L_S}]$  is the edge-cut such that  $X_S$  is the set of edges between  $A_{L_S}$  and  $B_{L_S}$ , and let  $f_{X_S}$  be the injection from  $V(S^+) - V(S)$  to  $X_S$  such that for every  $x \in V(S^+) - V(S)$ ,  $f_{X_S}(x)$  is the edge in  $X_S$  contained in  $\pi_E^{(S)}(x)$ . For each edge  $e$  of  $H_{\mathcal{P}}$  not contained in any member of  $\mathcal{Q}_{\mathcal{P}}$ , we define the following.

- Say  $e$  has one end in  $S_1 \in \mathcal{Q}_{\mathcal{P}}$  and one end in  $S_2 \in \mathcal{Q}_{\mathcal{P}}$ , it corresponds to a leaf  $e_1$  in  $S_1^+$  and a leaf  $e_2$  in  $S_2^+$ , and we define  $W_e = \{f_{X_{S_1}}(e_1), f_{X_{S_2}}(e_2)\}$ .
- We pick a set  $W'_e$  in  $\mathcal{X}'_E$  such that if  $e' \neq e''$ , then  $W'_e \neq W'_{e'}$ .
- Define  $(W_{e,1}, W_{e,2})$  to be a partition of  $W_e \cup W'_e$  into two sets of size two each containing exactly one element in  $W_e$ .

Let  $W$  be the union of  $W_{e,1}, W_{e,2}$  over all edges  $e$  of  $H_{\mathcal{P}}$  not contained in any member of  $\mathcal{Q}_{\mathcal{P}}$ . Note that those  $W_{e,1}, W_{e,2}$  form a partition of  $W$ . By Lemma 3.3, for each such edge  $e$  of  $H_{\mathcal{P}}$  and each  $i \in \{1, 2\}$ , there exists a tree  $T_{e,i}$  in  $G - Z_0$  such that these trees are edge-disjoint and  $E(T_{e,i}) \cap W = W_{e,i}$  for each  $e$ . We choose each  $T_{e,i}$  to be minimal, so  $E(T_{e,i})$  is disjoint from  $E(G[A_{L_S}])$  for all  $S \in \mathcal{Q}_{\mathcal{P}}$ . Since  $A_{L_{S'}} \cap A_{L_{S''}} = \emptyset$  for distinct  $S', S'' \in \mathcal{Q}_{\mathcal{P}}$



by Lemma 2.10, these trees together with the intersection of the image of  $\pi_E^{(S)}$  and  $G[A_{L_S}]$ , for each  $S \in \mathcal{Q}_{\mathcal{P}}$ , define an  $H_{\mathcal{P}}$ -immersion in  $G - Z_0$ , a contradiction.

Therefore, the above second statement hold for all shells  $\mathcal{P}$  of  $H$ . Note that  $|Z_{\mathcal{P}}| \leq (k^2(|\mathcal{Q}_{\mathcal{P}}| - |D_{\mathcal{P}}| + 2)h^2d_1^2)^{kd_1h+1} \leq (2k^3h^3d_1^2)^{kd_1h+1}$  for every shell  $\mathcal{P}$  of  $H$ . Let  $Z'_1$  be the union of  $Z_0$  and  $Z_{\mathcal{P}}$  for all shells  $\mathcal{P}$  of  $H$ . Note that there are at most  $h^h2^{hd_1}$  different shells, so  $|Z'_1| \leq \xi_0 + h^h2^{hd_1}(2k^3h^3d_1^2)^{kd_1h+1} \leq \xi_0 + (4k^3h^4d_1^2)^{kd_1h+1}$ . If  $G - Z'_1$  has no  $H$ -immersion, then we are done, so we assume that  $G - Z'_1$  has an  $H$ -immersion.

Note that every  $H$ -immersion in  $G - Z'_1$  is an  $H$ -immersion in  $G - Z_0$ . In addition, for every  $H$ -immersion  $L$  in  $G - Z_0$ , there exists a shell  $\mathcal{P}$  of  $H$  such that  $G[A_L]_G^+$  realizes  $S_H^+$  for every  $S \in \mathcal{Q}_{\mathcal{P}} - \{\{v\} : v \in D_{\mathcal{P}}\}$ . So for every  $H$ -immersion  $L$  in  $G - Z'_1$ , either  $Z'_1$  intersects some edge between  $A_L$  and  $B_L$ , or the set of edges between  $A_L$  and  $B_L$  is not free with respect to  $\mathcal{E} - Z'_1$ . Hence, for every  $H$ -immersion  $L$  in  $G - Z'_1$ , there exists  $[A'_L, B'_L] \in \mathcal{E} - Z'_1$  of order less than the order of  $[A_L, B_L]$  with  $A_L \subseteq A'_L$ . We then remove the components of  $G[A'_L] - Z'_1$  disconnected to the edges between  $A'_L$  and  $B'_L$  unless  $[A'_L, B'_L]$  has order zero in  $G - Z'_1$ . Note that  $A_{L'}$  still contains all branch vertices of  $L$  since  $H$  is connected. If it is possible to further delete at most  $2(h-1)kd_1$  edges from  $G - Z'_1$  to make some  $u \in U$  contained in  $A_u$  for some  $[A_u, B_u] \in \mathcal{E} - Z_1$  of order less than  $kd_1$ , then we put those edges into  $Z'_1$  to form a new set  $Z_1$  and remove all such vertices  $u$  from  $U$  and we replace  $A'_L$  by the union of  $A'_L$  and  $A_u$  for all such  $u$ . So we again can find for each  $u \in U$ , a new set of  $kd_1$  edges of  $G - Z_1$  incident with  $u$  such that the union of them is free with respect to  $\mathcal{E} - Z_1$ . Note that  $|Z_1| \leq |Z'_1| + 2(h-1)^2kd_1 \leq \xi_0 + (4k^3h^4d_1^2 + 1)^{kd_1h+1}$ .

So this process either decreases  $|U|$ , or does not increase  $|U|$  but decreases the maximum order of  $[A_L, B_L]$  among all  $H$ -immersions  $L$  in the remaining graph. Note that whenever  $|U|$  decreases by  $r$ , the order of  $[A'_L, B'_L]$  is increased by at most  $kd_1r$ , so its order is at most  $kd_1(|U| + h)$  if we repeat the process arbitrary times.

Therefore, by repeating the whole process at most  $d_1k(|U| + h)|U| \leq 2d_1kh^2$  times, there exists  $Z^* \subseteq E(G)$  with  $|Z^*| \leq \xi_0 + 2d_1kh^2 \cdot (4k^3h^4d_1^2 + 1)^{kd_1h+1} \leq \xi$  such that either  $G - Z^*$  has no  $H$ -immersion, or for every  $H$ -immersion  $L$  in  $G - Z^*$ , there exists  $[A_L^*, B_L^*] \in \mathcal{E} - Z^*$  of order zero such that all branch vertices of  $L$  are in  $A_L^*$ . We are done if the former happens. If the latter happens, the entire  $L$  is in  $G[A_L^*]$  since  $[A_L^*, B_L^*]$  has order zero. Define  $A^*$  to be the union of  $A_L^*$  over all  $H$ -immersions  $L$  in  $G - Z^*$ . Since

$\mathcal{E} - Z^*$  has order greater than zero,  $[A^*, V(G) - A^*] \in \mathcal{E} - Z^*$  has order zero and  $G[A^*]$  contains all  $H$ -immersions in  $G - Z^*$ . This completes the proof. ■

The following is the main lemma of this section.

**Lemma 3.9.** *For every graph  $H$  and for every positive integer  $k$ , there exist integers  $\theta, w, \xi$  such that the following holds. If  $G$  is a graph that does not contain  $k$  edge-disjoint  $H$ -immersions and  $\mathcal{E}$  is an edge-tangle in  $G$  of order at least  $\theta$  controlling a  $K_w$ -thorns, then there exist  $Z \subseteq E(G)$  with  $|Z| \leq \xi$  and  $[A, B] \in \mathcal{E} - Z$  of order zero in  $G - Z$  such that  $G[B] - Z$  contains no  $H$ -immersion.*

**Proof.** Let  $p$  be the number of components of  $H$ . We shall prove this lemma by induction on  $p$ . When  $p = 1$ , this lemma holds by taking  $\theta, w, \xi$  to be the numbers  $\theta, w, \xi$  mentioned in Lemma 3.8. So we may assume that  $p \geq 2$  and the lemma holds for every graph with less than  $p$  components.

Define  $\mathcal{F}_1$  to be the set of graphs that can be obtained from  $H$  by adding an edge between different components. Define  $\mathcal{F}_2$  to be the set of graphs that can be obtained from  $H$  by subdividing an edge and adding an edge between this new vertex and another component of  $H$ . Define  $\mathcal{F}_3$  to be the set of graphs that can be obtained from  $H$  by subdividing two edges in different components and either adding an edge between those two new vertices or identifying the two new vertices. Let  $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$ . Note that  $|\mathcal{F}| \leq |V(H)|^2 + |E(H)||V(H)| + 2|E(H)|^2 \leq 4|V(H)|^4$ . Since every graph in  $\mathcal{F}$  contains less than  $p$  components, by induction, for every graph  $F \in \mathcal{F}$  and for every positive integer  $k$ , there exists  $\theta_{F,k}, w_{F,k}, \xi_{F,k}$  such that the lemma holds. For every positive integer  $k$ , define  $\theta = \sum_{F \in \mathcal{F}} \theta_{F,k}$ ,  $w = \sum_{F \in \mathcal{F}} w_{F,k}$  and  $\xi = \sum_{F \in \mathcal{F}} \xi_{F,k}$ . We shall prove that the numbers  $\theta, w$  and  $\xi$  satisfy the lemma.

Let  $k$  be a positive integer, and let  $G$  be a graph that does not contain  $k$  edge-disjoint  $H$ -immersions and  $\mathcal{E}$  an edge-tangle in  $G$  of order at least  $\theta$  controlling an  $K_w$ -thorns. Note that for every  $F \in \mathcal{F}$ , an  $F$ -immersion in  $G - Z$  is an  $H$ -immersion in  $G - Z$ . By induction, for every  $F \in \mathcal{F}$ , there exist  $Z_F \subseteq E(G)$  with  $|Z_F| \leq \xi_F$  and  $[A_F, B_F] \in \mathcal{E} - Z_F$  of order zero in  $G - Z_F$  such that  $G[B_F] - Z_F$  contains no  $F$ -immersion. Define  $Z = \bigcup_{F \in \mathcal{F}} Z_F$  and  $[C, D] = [\bigcup_{F \in \mathcal{F}} A_F, \bigcap_{F \in \mathcal{F}} B_F]$ . So  $[C, D] \in \mathcal{E} - Z$  has order zero in  $G - Z$  and  $G[D] - Z$  contains no  $F$ -immersion for each  $F \in \mathcal{F}$ . Define  $[A, B]$  to be

the edge-cut in  $\mathcal{E} - Z$  of order zero in  $G - Z$  with the property that  $C \subseteq A$  such that  $A$  is maximal. Then  $G[B] - Z$  is connected by (E1) and (E2).

Suppose that  $G[B] - Z$  contains an  $H$ -immersion. Then for each  $i$  with  $1 \leq i \leq p$ ,  $G[B] - Z$  contains an  $H_i$ -immersion  $(\pi_V^{(i)}, \pi_E^{(i)})$ , where  $H_i$  is the  $i$ -th component of  $H$ , such that the images of  $\pi_V^{(1)}, \dots, \pi_V^{(p)}$  are pairwise disjoint and the images of  $\pi_E^{(1)}, \dots, \pi_E^{(p)}$  are pairwise edge-disjoint. Since  $G[B] - Z$  is connected, there exist distinct  $i, j$  and a path  $P$  in  $G[B] - Z$  from a vertex in the image of  $\pi_E^{(i)}$  to a vertex in the image of  $\pi_E^{(j)}$ . But it implies that  $G[B] - Z$  contains an  $F$ -immersion for some  $F \in \mathcal{F}$ , a contradiction. Therefore,  $G[B] - Z$  contains no  $H$ -immersion. ■

## 4 Edge-tangles in 4-edge-connected graphs

A  $m \times n$  grid is the graph with vertex-set  $\{1, 2, \dots, n\} \times \{1, 2, \dots, m\}$  and two vertices  $(x, y), (x', y')$  are adjacent if and only if  $|x - x'| + |y - y'| = 1$ . An  $H$ -immersion  $(\pi_v, \pi_e)$  is an  $H$ -subdivision if for every pair of edges  $e_1, e_2$  of  $H$ ,  $\pi_E(e_1) \cap \pi_E(e_2) \subseteq \pi_V(S)$ , where  $S$  is the set of the common ends of  $e_1, e_2$ . It is easy to see that every graph containing a  $r \times r$  grid-minor contains a  $r \times r$  wall-subdivision. We say that a tangle  $\mathcal{T}$  is induced by a  $r \times r$  wall-subdivision  $(\pi_V, \pi_E)$  if for every  $(A, B) \in \mathcal{T}$ ,  $E(B)$  intersects every path in the image of  $\pi_E$  of all edges of a row.

One corollary of the following restatement of [9, (2.3)] is that every graph with a tangle of large order contains a large grid minor and hence contains a subdivision of a large wall.

**Theorem 4.1.** [9, (2.3)] *Let  $\theta \geq 2$ , and let  $\mathcal{T}$  be a tangle in  $G$  of order at least  $20^{\theta^4(2\theta-1)}$ . If  $\mathcal{T}' \subseteq \mathcal{T}$  is a tangle of order  $\theta$ , then  $\mathcal{T}'$  is induced by a  $\theta \times \theta$  wall-subdivision.*

For every positive integer  $r$ , the *diagonal vertices* of the  $2r \times r$  wall are the vertices  $\{(2i - 1, i) : 1 \leq i \leq r\}$ .

**Lemma 4.2** ([1, (1.5)]). *For every  $g > 1$ , there exists  $b \geq 0$  such that the following holds. Let  $(\pi_V, \pi_E)$  be a wall-subdivision in a graph  $G$ , and let  $S$  be a subset of the image of  $\pi_V$  of the diagonal vertices of the wall such that for every pair of distinct vertices  $x, y$  in  $S$ ,  $G$  contains four edge-disjoint paths from  $x$  to  $y$ . If  $|S| \geq b$ , then there exists a  $g \times g$  grid-immersion  $(\pi'_V, \pi'_E)$  in  $G$  such that the image of  $\pi'_V$  is contained in  $S$ .*

In fact, in [1], Chudnovsky et al. proved that the grid-immersion  $(\pi'_V, \pi'_E)$  mentioned in Lemma 4.2 is a “strong immersion.” We omit the definition of strong immersions as we do not need this notion in the rest of the paper. But we remark that every  $H$ -strong immersion is an  $H$ -immersion. On the other hand, if we do not require  $(\pi'_V, \pi'_E)$  to be a strong immersion, we can strengthen Lemma 4.2 by showing that the mentioned wall-subdivision  $(\pi_V, \pi_E)$  can be replaced by a wall-immersion.

**Lemma 4.3.** *For every  $g > 1$ , there exists  $b \geq 0$  such that the following property holds. Let  $(\pi_V, \pi_E)$  be a wall-immersion in a graph  $G$ , and let  $S$  be a subset of the image of  $\pi_V$  of the diagonal vertices of the wall such that for every pair of distinct vertices  $x, y$  in  $S$ ,  $G$  contains four edge-disjoint paths from  $x$  to  $y$ . If  $|S| \geq b$ , then there exists a  $g \times g$  grid-immersion  $(\pi'_V, \pi'_E)$  in  $G$  such that the image of  $\pi'_V$  is contained in  $S$ .*

**Proof.** Let  $G'$  be the graph obtained from  $G$  by subdividing every edge once. Let  $H$  be the graph obtained from  $L(G')$  by adding a vertex  $u_v$  adjacent to every vertex in  $\text{cl}(v)$  for each  $v \in V(G) \subseteq V(G')$ . Then it is clear that  $H$  admits a wall-subdivision  $(\pi''_V, \pi''_E)$  such that  $\pi''_V(S) = \{u_s : s \in S\}$ . Note that for every non-loop edge  $e$  in  $G$  with ends  $x, y$ , there exists an edge in  $H$  with one end in  $V(\text{cl}(x))$  and one end in  $V(\text{cl}(y))$ , and we also denote this edge in  $H$  as  $e$ . Since every wall does not contain a loop, the image of  $\pi''_E$  of each edge is path in  $H$ .

For every pair of distinct vertices  $x, y$  of  $S$ , there exist four edge-disjoint paths  $P_1, P_2, P_3, P_4$  in  $G$  from  $x$  to  $y$ , so it is clear that there exist four paths  $Q_1, Q_2, Q_3, Q_4$  in  $H$  from  $u_x$  to  $u_y$  such that  $E(Q_i)$  contains  $E(P_i)$  for  $1 \leq i \leq 4$ . If we choose those paths  $Q_1, \dots, Q_4$  such that the sum of their length is minimum, then  $Q_1, \dots, Q_4$  are pairwise edge-disjoint. Therefore, by Lemma 4.2,  $H$  admits a  $g \times g$  grid-immersion  $(\pi'''_V, \pi'''_E)$  such that the image of  $\pi'''_V$  of the grid is contained in  $\{u_s : s \in S\}$ . Then by contracting  $\{u_s\} \cup \text{cl}(s)$  in  $H$  for each  $s \in S$  into  $s$  and contracting  $\text{cl}(v)$  in  $H$  into  $v$  for each  $v \in V(G) - S$ ,  $G$  admits a  $g \times g$  grid-immersion  $(\pi^*_V, \pi^*_E)$  such that the image of  $\pi^*_V$  is  $\{s \in S : u_s \text{ is in the image of } \pi'''_V\}$ . This proves the lemma. ■

The following lemma shows that every 4-edge-connected graph with an edge-tangle induced by an immersion of a large wall has an edge-tangle controlling a complete graph-thorns.

**Lemma 4.4.** *For every positive integers  $\theta$  and  $t$ , there exists a positive integer  $w$  such that the following holds. If  $G$  is a 4-edge-connected graph and*

$\mathcal{E}$  is an edge-tangle in  $G$  of order  $w$  induced by a  $w \times w$  wall-immersion, then there exists an edge-tangle  $\mathcal{E}' \subseteq \mathcal{E}$  of order at least  $\theta$  in  $G$  controlling a  $K_t$ -thorns.

**Proof.** We may assume that  $\theta \geq 2t$ . Let  $w$  be the number  $b$  mentioned in Lemma 4.3 by taking  $g = 2\theta$ . Denote the  $w \times w$ -wall by  $W$  and denote the  $2\theta \times 2\theta$ -grid by  $R$ . Let  $S$  be the set of diagonal vertices of  $W$ . Let  $(\pi_V, \pi_E)$  be a  $W$ -immersion in  $G$  inducing  $\mathcal{E}$ . By Lemma 4.3,  $G$  admits a  $R$ -immersion  $(\pi'_V, \pi'_E)$  such that  $\pi'_V(V(R))$  is contained in  $\pi_V(S)$ . Define  $\mathcal{E}'$  to be the collection of all edge-cuts  $[A, B]$  of  $G$  of order less than  $\theta$  such that  $B$  contains the image of  $\pi'_V$  of all vertices of a column and a row of  $R$ . Since  $R$  contains a  $2\theta \times 2\theta$ -wall as a subgraph,  $\mathcal{E}'$  is an edge-tangle in  $G$  of order  $\theta$  by Lemmas 2.7 and 2.9. For every  $1 \leq i \leq t$ , define  $\alpha(v_i)$  to be the union of the image of  $\pi'_E$  of the  $i$ -th column and the  $i$ -th row of  $R$ , where we write  $V(K_t) = \{v_j : 1 \leq j \leq t\}$ . So  $\alpha$  is a  $K_t$ -thorns.

We claim that  $\mathcal{E}'$  controls  $\alpha$ . Suppose that there exist  $[A, B] \in \mathcal{E}'$  with order less than  $t$  and  $v \in V(K_t)$  such that  $V(\alpha(v)) \cap B = \emptyset$ . Since  $[A, B] \in \mathcal{E}'$ ,  $B$  contains the image of  $\pi'_V$  of a column of  $R$ . Since  $\alpha(v)$  intersects the image of  $\pi'_V$  of each column,  $B \cap V(\alpha(v)) \neq \emptyset$ , a contradiction. Hence  $\mathcal{E}'$  controls a  $K_t$ -thorns  $\alpha$ .

It suffices to prove that  $\mathcal{E}' \subseteq \mathcal{E}$  to complete the proof. Let  $[A, B] \in \mathcal{E}'$ , so the order of  $[A, B]$  is less than  $\theta$ . Since  $\pi'_V(V(R)) \subseteq \pi_V(S)$  and  $B$  contains the image of  $\pi'_V$  of all vertices of a row of  $R$ ,  $B$  contains at least  $2\theta$  vertices in  $\pi_V(S)$ . Suppose that  $[B, A] \in \mathcal{E}$ , then  $A$  contains the image of  $\pi_V$  of all vertices a column of  $W$ . So the order of  $[A, B]$  is at least  $2\theta$ , a contradiction. This proves that  $[A, B] \in \mathcal{E}$ . ■

**Lemma 4.5.** For every positive integers  $\theta$  and  $d$ , there exists an integer  $w$  such that if  $\mathcal{E}$  is an edge-tangle in a graph  $G$  of order at least  $w$ , then either there exists  $v \in V(G)$  incident with at least  $d$  edges in  $G$  such that  $v \in B$  for every  $[A, B] \in \mathcal{E}_\theta$ , or  $\mathcal{E}_\theta$  is induced by a  $\theta \times \theta$  wall-immersion, where  $\mathcal{E}_\theta$  is the edge-tangle in  $G$  of order  $\theta$  such that  $\mathcal{E}_\theta \subseteq \mathcal{E}$ .

**Proof.** Let  $\theta' = 4^\theta \theta^{2\theta+1} d^{2\theta}$  and  $w = 20^{64\theta'^5}$ . Denote the  $\theta' \times \theta'$  wall by  $W$ . Assume that  $\mathcal{E}$  is an edge-tangle in  $G$  of order at least  $w$ . By Lemma 2.3,  $\overline{\mathcal{E}}$  is a tangle of order at least  $w/3 - 1 \geq 20^{(2\theta')^4(4\theta'-1)}$  in  $L(G)$ . For every integer  $t$ , let  $\overline{\mathcal{E}}_t$  be the tangle in  $L(G)$  of order  $t$  with  $\overline{\mathcal{E}}_t \subseteq \overline{\mathcal{E}}$ . By Theorem 4.1,  $\overline{\mathcal{E}}_{\theta'}$  is induced by a  $W$ -subdivision  $(\pi_V, \pi_E)$ .

Assume that there exists  $v \in V(G)$  such that  $\text{cl}(v)$  contains at least  $(2\theta d)^2$  vertices in  $\pi_V(V(W))$ . So  $|\text{cl}(v)| \geq (2\theta d)^2$  and  $v$  is incident with at least  $(2\theta d)^2 \geq d$  edges in  $G$ . Suppose that there exists an edge-cut  $[A, B] \in \mathcal{E}_\theta$  such that  $v \in A$ . We may assume that the order of  $[A, B]$  is as small as possible. Define  $B'$  to be the subgraph of  $L(G)$  induced by the vertices of  $L(G)$  corresponding to the edges of  $G$  incident with vertices in  $B$ , and define  $(A', B')$  to be the separation of  $L(G)$  such that  $V(A') \cap V(B')$  consists of the vertices of  $L(G)$  corresponding to the edges with one end in  $A$  and one end in  $B$ , and subject to that,  $E(A)$  is maximal. Since the order of  $[A, B]$  is minimal,  $(A', B')$  is normalized. Since  $v$  is incident with at least  $(2\theta d)^2 > \theta$  edges in  $G$ , there exists an edge of  $G$  incident with  $v$  having both ends in  $A$ . So  $\text{cl}(v) \subseteq V(A')$  and  $[A, B]$  is the partner of  $(A', B')$ . Therefore,  $(A', B') \in \overline{\mathcal{E}}$  and  $A'$  contains  $(2\theta d)^2$  vertices in  $\pi_V(V(W))$ . The former implies that  $V(B')$  contains the image of  $\pi_V$  of all vertices of a row and a column of  $W$ ; the latter implies that  $V(A')$  contains the image of  $\pi_V$  of some vertices in either at least  $2\theta d$  rows of  $W$  or at least  $\theta d$  columns of  $W$ . Hence, the order of  $(A', B')$  is at least  $\theta d \geq \theta$ . This implies that the order of  $[A, B]$  is at least  $\theta$ , a contradiction. So  $v$  is a vertex of  $G$  incident with at least  $(2\theta d)^2 \geq d$  edges in  $G$  such that  $v \in B$  for every edge-cut  $[A, B] \in \mathcal{E}_\theta$ . Therefore, we may assume that for every  $v \in V(G)$ ,  $\text{cl}(v)$  contains at most  $(2\theta d)^2$  vertices in  $\pi_V(V(W))$ .

Define  $W_0 = W$ ,  $t_0 = \theta'$ , and  $(\pi_V^{(0)}, \pi_E^{(0)}) = (\pi_V, \pi_E)$ . For  $i \geq 0$ , define  $t_{i+1} = t_i/(4\theta^2 d^2)$ , define  $W_{i+1}$  to be the  $\theta' \times t_{i+1}$  wall, and define  $(\pi_V^{(i+1)}, \pi_E^{(i+1)})$  to be the  $W_{i+1}$ -subdivision in  $L(G)$  such that  $\pi_V^{(i+1)}(V(W_{i+1})) \subseteq \pi_V^{(i)}(V(W_i))$  and  $\bigcup_{e \in E(W_{i+1})} \pi_E^{(i+1)}(e) \subseteq \bigcup_{e \in E(W_i)} \pi_E^{(i)}(e)$  such that no distinct vertices  $x, y$  in the first  $i$ -th rows of  $W_{i+1}$  are contained in the same  $\text{cl}(v)$  for some  $v \in V(G)$ . Note that each  $W_i$  exists as for every  $v \in V(G)$ ,  $\text{cl}(v)$  contains at most  $(2\theta d)^2$  vertices in  $\pi_V(V(W))$ . So  $(\pi_V^{(\theta)}, \pi_E^{(\theta)})$  is a  $\theta' \times t_\theta$  wall-subdivision in  $L(G)$  such that for every  $v \in V(G)$ ,  $\text{cl}(v)$  contains at most one vertex in  $\pi_V^{(\theta)}(V(W_\theta))$ . Note that  $t_\theta \geq \theta'/(4\theta^2 d^2)^\theta \geq \theta$ , so there exists a  $\theta \times \theta$ -wall subdivision  $(\pi_V^*, \pi_E^*)$  in  $L(G)$  such that for every  $v \in V(G)$ ,  $\text{cl}(v)$  contains at most one vertex in the image of  $\pi_V^*$ .

Denote the  $\theta \times \theta$  wall by  $W'$ . Now we define a  $W'$ -immersion  $(\pi'_V, \pi'_E)$  in  $G$ . Define  $\pi'_V$  to be the function that maps each vertex  $x$  of  $W'$  to the vertex  $v$  of  $G$  such that  $\pi_V^*(x) \in \text{cl}(v)$ ; define  $\pi'_E$  to be the function that maps each edge  $e$  of  $W'$  to the path in  $G$  with the edge-set equal to the vertex-set of  $\pi_E^*(e)$ . It is clear that  $(\pi'_V, \pi'_E)$  is a  $W'$ -immersion in  $G$ . It is sufficient to

show that  $(\pi'_V, \pi'_E)$  induces  $\mathcal{E}_\theta$ . Let  $[A, B] \in \mathcal{E}_\theta$ . Let  $\mathcal{E}''$  be the edge-tangle induced by  $(\pi'_V, \pi'_E)$ . We shall prove that  $[A, B] \in \mathcal{E}''$ .

We first assume that the only isolated vertices in  $G[A]$  and  $G[B]$  are the isolated vertices of  $G$ . Define  $B'$  to be the subgraph of  $L(G)$  induced by the vertices of  $L(G)$  corresponding to the edges of  $G$  incident with vertices in  $B$ , and define  $(A', B')$  to be the separation of  $L(G)$  such that  $V(A') \cap V(B')$  consists of the vertices of  $L(G)$  corresponding to the edges with one end in  $A$  and one end in  $B$ , and subject to that,  $E(A)$  is maximal. Since for every non-isolated vertex  $v$  of  $G$ , it has a neighbor in the same side of the edge-cut, every vertex in  $V(A') \cap V(B')$  is adjacent to a vertex in  $V(A') - V(B')$  and a vertex in  $V(B') - V(A')$ . So  $(A', B')$  is normalized and  $[A, B]$  is the partner of  $(A', B')$ . Since  $[A, B] \in \mathcal{E}_\theta$ ,  $(A', B') \in \overline{\mathcal{E}}_\theta$ . Since  $\overline{\mathcal{E}}_{\theta'}$  is induced by a  $W$ -subdivision  $(\pi_V, \pi_E)$ ,  $E(B')$  intersects every path in the image of  $\pi_E$  of all edges of a row of  $W$ . Since the order of  $(A', B')$  is less than  $\theta$  and  $W$  is a  $\theta' \times \theta'$  wall,  $V(B') - V(A')$  contains at least  $\theta' - \theta$  vertices in the image of  $\pi_V$  of the vertices of a row of  $W$ . Hence,  $B$  contains at least  $(\theta' - \theta)/(4\theta^2 d^2)$  vertices  $v$  of  $G$  such that the union of  $\text{cl}(v)$  over such vertices  $v$  contains at least  $\theta' - \theta$  vertices in the image of  $\pi_V$  of the vertices of a row of  $W$ . Since  $\mathcal{E}''$  has order  $\theta$ , either  $[A, B]$  or  $[B, A]$  is in  $\mathcal{E}''$ . Suppose that  $[B, A] \in \mathcal{E}''$ . So  $A$  contains the image of  $\pi'_V$  of all vertices of a row of  $W'$ . Hence,  $V(A')$  contains  $\theta$  vertices  $v$  such that the union of  $\text{cl}(v)$  over all such vertices  $v$  contains at least  $\theta$  vertices in the image of  $\pi_V$  of the vertices of a row of  $W$ . However, Since  $(\pi_V, \pi_E)$  is a  $W$ -subdivision, the order of  $(A', B')$  is at least  $\min\{\theta' - \theta, \theta\} > \theta$ , a contradiction. Therefore,  $[A, B] \in \mathcal{E}''$ .

Now we assume that some isolated vertex  $v$  of  $G[A]$  or  $G[B]$  is not an isolated vertex of  $G$ . In other words, there exists a vertex of  $v$  of  $G$  such that all edges incident with  $v$  are between  $A$  and  $B$ . In particular,  $v$  has degree less than  $\theta$ . So  $[\{v\}, V(G) - \{v\}] \in \mathcal{E}_\theta \cap \mathcal{E}''$  by (E1) and (E3). We may assume that  $[A, B]$  is an edge-cut such that the number of such vertices  $v$  is as small as possible. If  $v \in A$ , then  $[A - \{v\}, B \cup \{v\}] \in \mathcal{E}_\theta$  since  $\mathcal{E}_\theta$  satisfies (E2). And the minimality of  $[A, B]$  implies that  $[A - \{v\}, B \cup \{v\}] \in \mathcal{E}'' \cap \mathcal{E}_\theta$ . Since  $\mathcal{E}''$  satisfies (E2) and  $A \cap (B \cup \{v\}) \cap (V(G) - \{v\}) = \emptyset$ , we have that  $[A, B] \in \mathcal{E}''$ . Similarly, if  $v \in B$ , then  $[A \cup \{v\}, B - \{v\}] \in \mathcal{E}_\theta$  since  $\mathcal{E}_\theta$  satisfies (E2), otherwise  $[B - \{v\}, A \cup \{v\}]$ ,  $[A, B]$ ,  $[\{v\}, V(G) - \{v\}]$  are three edge-cuts in  $\mathcal{E}_\theta$  such that  $(A \cup \{v\}) \cap B \cap (V(G) - \{v\}) = \emptyset$ . And the minimality of  $[A, B]$  implies that  $[A \cup \{v\}, B - \{v\}] \in \mathcal{E}'' \cap \mathcal{E}_\theta$ . Since  $\mathcal{E}''$  satisfies (E2),  $[A, B] \in \mathcal{E}''$ . This proves that  $\mathcal{E}_\theta$  is induced by  $(\pi'_V, \pi'_E)$ . ■

**Theorem 4.6.** *For every positive integers  $k$  and  $\theta$ , there exists a positive integer  $w$  such that if  $G$  is a 4-edge-connected graph and  $\mathcal{E}$  is an edge-tangle in  $G$  of order at least  $w$ , then  $\mathcal{E}_\theta$  controls a  $K_k$ -thorns, where  $\mathcal{E}_\theta$  is the edge-tangle in  $G$  of order  $\theta$  such that  $\mathcal{E}_\theta \subseteq \mathcal{E}$ .*

**Proof.** Let  $k$  and  $\theta$  be positive integers. We may assume that  $\theta > k$ . For every integer  $t$  and for every edge-tangle  $\mathcal{E}$  in a graph, let  $\mathcal{E}_t$  be the edge-tangle in  $G$  of order  $t$  such that  $\mathcal{E}_t \subseteq \mathcal{E}$ . By Lemma 4.4, there exists a positive integer  $w_1$  such that if  $G$  is a 4-edge-connected graph and  $\mathcal{E}'$  is an edge-tangle in  $G$  of order at least  $w_1$  induced by a  $w_1 \times w_1$  wall-immersion, then  $\mathcal{E}'_\theta$  controls a  $K_k$ -thorns. Note that it implies that  $w_1 \geq \theta$ . By Lemma 4.5, there exists a positive integer  $w$  such that if  $G$  is a 4-edge-connected graph and  $\mathcal{E}$  is an edge-tangle in  $G$  of order at least  $w$ , then either there exists a vertex  $v \in V(G)$  incident with at least  $k$  edges of  $G$  such that  $v \in B$  for every  $[A, B] \in \mathcal{E}_{w_1}$ , or  $\mathcal{E}_{w_1}$  is induced by a  $w_1 \times w_1$  wall-immersion.

Now let  $G$  be a 4-edge-connected graph and  $\mathcal{E}$  an edge-tangle in  $G$  of order at least  $w$ . Let  $\mathcal{E}' = \mathcal{E}_{w_1}$ . If  $\mathcal{E}'$  is induced by a  $w_1 \times w_1$  wall-immersion, then  $\mathcal{E}'_\theta$  controls a  $K_k$ -thorns, and hence  $\mathcal{E}_\theta$  controls a  $K_k$ -thorns. So we may assume that such a vertex  $v$  incident with at least  $k$  edges exists. Define  $\alpha$  to be a  $K_k$ -thorns such that  $\alpha(h)$  is an edge of  $G$  incident with  $v$  for each  $h \in V(K_k)$ . We shall prove that  $\mathcal{E}_\theta$  controls  $\alpha$ . Let  $[A, B] \in \mathcal{E}_\theta$  with order less than  $k$ . So  $[A, B] \in \mathcal{E}_{w_1}$ . Hence,  $v \in B \cap V(\alpha(h))$  for every  $h \in V(K_k)$ . Therefore,  $\mathcal{E}_\theta$  controls a  $K_k$ -thorns. ■

## 5 Erdős-Pósa property

We say that a graph  $G$  is *nearly 4-edge-connected* if  $G$  is connected and for every edge-cut of  $G$  of order less than four, the edges between  $A$  and  $B$  are the parallel edges with the same ends.

**Lemma 5.1.** *If  $G$  is a nearly 4-edge-connected graph, then there exist a tree  $T$  and a partition  $\{X_t : t \in V(T)\}$  of  $V(G)$  such that the following hold.*

1. *For every  $t \in V(T)$ ,  $G[X_t]$  either consists of the single vertex or is 4-edge-connected.*
2. *If there is an edge of  $G$  with one end in  $X_{t_1}$  and one end in  $X_{t_2}$  for some distinct  $t_1, t_2 \in V(T)$ , then  $t_1$  is adjacent to  $t_2$  in  $T$ .*



3. For every edge  $t_1t_2$  of  $T$ , there are at most three edges with one end in  $X_{t_1}$  and one end in  $X_{t_2}$ , and those edges are the parallel edges of the same ends.

**Proof.** We prove this lemma by induction on  $|V(G)|$ . If  $G$  is 4-edge-connected or consists of the single vertex, then we are done by taking the tree on one vertex and the partition of  $V(G)$  with one part. This proves the base case and we may assume that the lemma holds for every nearly 4-edge-connected graph on less than  $|V(G)|$  vertices. So we may assume that there exists an edge-cut  $[A, B]$  of  $G$  of order less than four such that the edges between  $A$  and  $B$  are the parallel edges with the same ends  $u, v$ , say  $u \in A$  and  $v \in B$ . Clearly,  $G[A]$  and  $G[B]$  are nearly 4-edge-connected. By induction, there exist trees  $T_A, T_B$ , a partition  $\{Y_t : t \in V(T_A)\}$  of  $A$  and a partition  $\{Z_t : t \in V(T_B)\}$  of  $B$  satisfying the three properties mentioned in the lemma. Let  $t_u \in V(T_A)$  and  $t_v \in V(T_B)$  such that  $u \in X_{t_u}$  and  $v \in X_{t_v}$ . Define  $T$  to be the tree obtained from the union of  $T_A$  and  $T_B$  by adding the edge  $t_ut_v$ . For every  $t \in V(T)$ , define  $X_t = Y_t$  if  $t \in V(T_A)$ , and  $X_t = Z_t$  if  $t \in V(T_B)$ . Then  $T$  and the partition  $\{X_t : t \in V(T)\}$  of  $V(G)$  satisfy the three properties mentioned in the lemma. ■

An *isolated vertex* in a graph is a vertex of degree zero. Now we are ready to address the Erdős-Pósa property.

**Theorem 5.2.** *For every graph  $H$  with no isolated vertices, there exist functions  $f : (\mathbb{N} \cup \{0\})^2 \rightarrow \mathbb{N}$  and  $g : \mathbb{N} \rightarrow \mathbb{N}$  such that for every nearly 4-edge-connected graph  $G$ , for every positive integer  $k$  and for every  $S \subseteq V(G)$  containing no vertex of degree at least  $g(k)$ , either  $G$  contains  $k$  edge-disjoint  $H$ -immersions, or  $G - S$  contains at least  $k - |S|$  edge-disjoint  $H$ -immersions, or there exists  $Z \subseteq E(G)$  with  $|Z| \leq f(k, |S|)$  such that  $G - Z$  does not contain an  $H$ -immersion.*

**Proof.** Let  $H$  be a fixed graph with no isolated vertices in this proof. Denote  $|V(H)|$  by  $h$  and the maximum degree of  $H$  by  $d$ . We shall prove this theorem by induction on  $|E(H)|$ . If  $H$  contains only one edge, then  $H = K_2$ , so every graph  $G$  with at least  $k$  non-loop edges contains  $k$  edge-disjoint  $H$ -immersions, and for every graph  $G$  with less than  $k$  non-loop edges, there exists  $Z \subseteq E(G)$  with  $|Z| \leq k$  such that  $G - Z$  has no non-loop edge and has no  $H$ -immersion. This proves the base case of the induction. We assume that the theorem is true for every graph  $H'$  without isolated vertices with

$|E(H')| < |E(H)|$  and denote the corresponding functions  $f$  and  $g$  by  $f_{H'}$  and  $g_{H'}$ , respectively. In particular, every nearly 4-edge-connected graph either contains  $k$  edge-disjoint  $H'$ -immersions or has a hitting set for  $H'$ -immersions of size at most  $f_{H'}(k, 0)$ . Define  $f'_H$  to be the function from  $\mathbb{N} \cup \{0\}$  to  $\mathbb{R}$  such that  $f'_H(k) = \sum f_{H'}(k, 0)$  for every nonnegative integer  $k$ , where the sum is taken over all graphs  $H'$  with no isolated vertices having less edges than  $H$ .

For every positive integer  $n$ , define  $\theta_n, w_n, \xi_n$  to be the numbers  $\theta, w, \xi$ , respectively, mentioned in Lemma 3.9 by taking  $H = H$  and  $k = n$ . Note that  $\theta_n \geq \xi_n$  for every  $n$ . For every positive integer  $n$ , define  $g(n) = w + 3nhd$ , where  $w$  is the number  $w$  mentioned in Theorem 4.6 by taking  $k = w_n$  and  $\theta = \theta_n$ . Define  $f(0, 0) = f(m, n) = 0$  for every nonnegative integers  $m, n$  with  $m < n$ ;  $f(m, m) = 2f(m-1, 0) + 2mg(m) + (2^{h+1} - 8)f'_H(mg(m))$  for every positive integer  $m$ ;  $f(m, n) = 2f(m, n+1) + 2g(m) + (2^{h+1} - 8)f'_H(mg(m))$  for every integers  $m, n$  with  $m > n \geq 0$ . Note that  $f(m, n) \geq 2mg(m) + (2^{h+1} - 8)f'_H(mg(m))$  for every  $m > 0$ .

We shall prove that the functions  $f$  and  $g$  defined above satisfy the conclusion of the theorem by induction on  $(k - |S|, -|S|)$ . Let  $G$  be a nearly 4-edge-connected graph that does not contain  $k$  edge-disjoint  $H$ -immersions, and let  $S \subseteq V(G)$  containing no vertex of degree at least  $g(k)$ . The theorem is obvious when  $k - |S| \leq 0$ . This proves the base case. So we may assume that  $k - |S| \geq 1$  and  $f$  satisfies the conclusion for every pair  $(k', S')$  with  $(k' - |S'|, -|S'|) < (k - |S|, -|S|)$ , where the pairs are compared by the lexicographic order. Suppose that  $G$  is a counterexample. That is,  $G - S$  does not contain at least  $k - |S|$  edge-disjoint  $H$ -immersions, but it is impossible to delete at most  $f(k, |S|)$  edges from  $G$  to kill all  $H$ -immersions in  $G$ .

For every edge-cut  $[A, B]$  of  $G$ , define  $G_A$  and  $G_B$  to be the graph obtained from  $G$  by identifying  $B$  and  $A$  into one new vertex  $v_B$  and  $v_A$ , respectively, and deleting the resulting loops. Define  $S_A = (S \cap A) \cup \{v_B\}$  and  $S_B = (S \cap B) \cup \{v_A\}$ . Note that the degree of  $v_B$  in  $G_A$  and the degree of  $v_A$  in  $G_B$  are the order of  $[A, B]$ .

**Claim 1:** If there exist  $W \subseteq V(G)$ , an edge-cut  $[A, B]$  of  $G$  of order less than  $g(k)$  and  $Z_0 \subseteq E(G)$  containing all edges between  $A$  and  $B$  such that  $G[A] - (W \cup Z_0)$  and  $G[B] - (W \cup Z_0)$  do not contain  $H$ -immersions, then there exists  $Z$  with  $Z_0 \subseteq Z \subseteq E(G)$  and  $|Z| \leq |Z_0| + (2^h - 4)f'(kg(k))$  such that  $G - (W \cup Z)$  has no  $H$ -immersion.

**Proof of Claim 1:** If  $H$  is connected, then every  $H$ -immersion in  $G - Z_0$  must be in  $G[A]$  or  $G[B]$  as  $Z_0$  contains all edges between  $A$  and  $B$ . So we are done by taking  $Z = Z_0$ . Now we assume that  $H$  is not connected.

Let  $H_1, H_2, \dots, H_p$  be the components of  $H$ , where  $p \geq 2$ . For every set  $I$  with  $\emptyset \subset I \subset [p]$ , define  $Q_I$  to be the disjoint union of  $H_i$  for all  $i \in I$ . Since  $H$  has no isolated vertices, every  $H$ -immersion intersecting  $S_A$  (or  $S_B$ ) must intersect an edge incident with a vertex in  $S_A$  (or  $S_B$ ). Since every vertex in  $S_A$  has degree at most  $g(k) - 1$  in  $G_A$ , for every  $I$  with  $\emptyset \subset I \subset [p]$ , if  $G_A$  (or  $G_B$ , respectively) contains at least  $k + |S_A|(g(k) - 1)$  (or  $k + |S_B|(g(k) - 1)$ , respectively) edge-disjoint  $Q_I$ -immersions, then  $G[A] - S_A$  (or  $G[B] - S_B$ , respectively) contains  $k$  edge-disjoint  $Q_I$ -immersions. Since  $G$  has no  $k$  edge-disjoint  $H$ -immersions, for every  $I$  with  $\emptyset \subset I \subset [p]$ , either  $G_A$  does not contain  $k + |S_A|(g(k) - 1)$  edge-disjoint  $Q_I$ -immersions, or  $G_B$  does not contain  $k + |S_B|(g(k) - 1)$  edge-disjoint  $Q_{[p]-I}$ -immersions. As  $k - |S| \geq 1$ ,  $\max\{|S_A|, |S_B|\} \leq |S| + 1 \leq k$ . So either  $G_A$  does not contain  $kg(k)$  edge-disjoint  $Q_I$ -immersions, or  $G_B$  does not contain  $kg(k)$  edge-disjoint  $Q_{[p]-I}$ -immersions. Each  $Q_I$  has less edges than  $H$  and has no isolated vertices. So for every  $I$  with  $\emptyset \subset I \subset [p]$ , there exists  $Z_I \subseteq E(G)$  with  $|Z_I| \leq f_{Q_I}(kg(k), 0) + f_{Q_{[p]-I}}(kg(k), 0) \leq 2f'_H(kg(k))$  such that either  $G_A - Z_I$  has no  $Q_I$ -immersion or  $G_B - Z_I$  has no  $Q_{[p]-I}$ -immersion. Define  $Z = Z_0 \cup \bigcup_{\emptyset \subset I \subset [p]} Z_I$ . Note that  $|Z| \leq |Z_0| + (2^p - 2) \cdot 2f'_H(kg(k)) \leq |Z_0| + (2^h - 4)f'_H(kg(k))$ , since  $2p \leq h$ .

Suppose that  $G - (W \cup Z)$  contains an  $H$ -immersion. Since  $Z$  contains all edges between  $A$  and  $B$ , and  $G[A] - (W \cup Z_0)$  and  $G[B] - (W \cup Z_0)$  contains no  $H$ -immersion, respectively, there exists  $I$  with  $\emptyset \subset I \subset [p]$  such that  $G[A] - Z$  contains a  $Q_I$ -immersion and  $G[B] - Z$  contains a  $Q_{[p]-I}$ -immersion, contradicting the existence of  $Z_I$ . This proves the claim.  $\square$

**Claim 2:** There exists no edge-cut  $[A, B]$  of  $G$  of order less than  $g(k)$  such that each  $G[A] - S$  and  $G[B] - S$  contains an  $H$ -immersion.

**Proof of Claim 2:** Suppose that there exists an edge-cut  $[A, B]$  of  $G$  of order less than  $g(k)$  such that each  $G[A] - S$  and  $G[B] - S$  contains an  $H$ -immersion. Recall that  $\deg_{G_A}(v_B) < g(k)$  and  $\deg_{G_B}(v_A) < g(k)$ . Since each  $G[A]$  and  $G[B]$  contains an  $H$ -immersion,  $G_A$  and  $G_B$  contain at least two vertices. If  $G_A$  is not nearly 4-edge-connected, then there exists an edge-cut  $[X, Y]$  of  $G_A$  of order at most three such that  $v_B \in Y$  and the edges between  $X$  and  $Y$  are not parallel edges with the same ends. But then  $[X, (Y - \{v_B\}) \cup B]$  is an edge-cut of  $G$  of order at most three and the edges in between are not parallel edges with the same ends, a contradiction. So  $G_A$  is nearly 4-edge-connected. Since  $G[B] - S$  contains an  $H$ -immersion,  $G[A]$  does not contain  $k - 1$  edge-disjoint  $H$ -immersions and  $G[A] - S_A$  does not contain  $k - |S| - 1$  edge-disjoint

$H$ -immersions. If  $S \cap B \neq \emptyset$ , then  $|S_A| \leq |S|$  and  $G_A - S_A$  does not contain  $k - |S_A| - 1$  edge-disjoint  $H$ -immersions; if  $S \cap B = \emptyset$ , then  $|S_A| = |S| + 1$  and  $G_A - S_A$  does not contain  $k - |S_A| = k - |S| - 1$  edge-disjoint  $H$ -immersions. By induction, there exists  $Z_A \subseteq E(G_A)$  with  $|Z_A| \leq f(k - 1, |S_A|)$  for the former case and  $|Z_A| \leq f(k, |S| + 1)$  for the latter case, such that  $G_A - Z_A$  contains no  $H$ -immersion. In either case,  $|Z_A| \leq f(k, |S| + 1)$ . Similarly, there exists  $Z_B \subseteq E(G_B)$  with  $|Z_B| \leq f(k, |S| + 1)$  such that  $G_B - Z_B$  contains no  $H$ -immersion. Note that  $Z_A$  and  $Z_B$  are subsets of  $E(G)$ . Define  $Z_0 = Z_A \cup Z_B \cup Z'$ , where  $Z'$  is the set of edges of  $G$  with one end in  $A$  and one end in  $B$ . Note that  $|Z_0| \leq 2f(k, |S| + 1) + g(k)$ .

$G[A] - Z_0$  and  $G[B] - Z_0$  have no  $H$ -immersion and  $Z_0$  contains all edges between  $A$  and  $B$ , so by applying Claim 1 by taking  $W = \emptyset$ , there exists  $Z$  with  $Z_0 \subseteq Z \subseteq E(G)$  and  $|Z| \leq |Z_0| + (2^h - 4)f'(kg(k)) \leq f(k, |S|)$  such that  $G - Z$  has no  $H$ -immersion. This contradicts the assumption that  $G$  is a counterexample.  $\square$

**Claim 3:** For every edge-cut  $[A, B]$  of  $G$  of order less than  $g(k)$ , exactly one of  $G[A] - S$  or  $G[B] - S$  contains an  $H$ -immersion.

**Proof of Claim 3:** Suppose to the contrary. So there exists an edge-cut  $[A, B]$  of  $G$  of order less than  $g(k)$  such that  $G[A] - S$  and  $G[B] - S$  do not contain  $H$ -immersions by Claim 2. Applying Claim 1 by taking  $W = S$  and  $Z_0$  to be the set of the edges between  $A$  and  $B$ , we obtain  $Z \subseteq E(G)$  with  $|Z| \leq g(k) + (2^h - 4)f'_H(kg(k))$  such that  $G - (S \cup Z)$  has no  $H$ -immersion.

Let  $Z'$  be the union of  $Z$  and the set of edges incident with vertices in  $S$ . Since  $|S| \leq k - 1$ ,  $|Z'| \leq |Z| + (k - 1)(g(k) - 1) \leq kg(k) + (2^h - 4)f'(kg(k)) \leq f(k, |S|)$ . Since  $H$  has no isolated vertices,  $G - Z'$  has no  $H$ -immersion, a contradiction.  $\square$

Define  $\mathcal{E}$  to be the collection of edge-cuts of  $G$  such that  $[A, B] \in \mathcal{E}$  if and only if it has order less than  $g(k)$  and  $G[B] - S$  contains an  $H$ -immersion.

**Claim 4:**  $\mathcal{E}$  is an edge-tangle in  $G$  of order  $g(k)$ .

**Proof of Claim 4:** Claim 3 implies that  $\mathcal{E}$  satisfies (E1). Suppose that there exist edge-cuts  $[A_1, B_1], [A_2, B_2], [A_3, B_3] \in \mathcal{E}$  with  $B_1 \cap B_2 \cap B_3 = \emptyset$ . Then  $B_3 \subseteq A_1 \cup A_2$ . Since  $G[B_3 - A_1]$  is a subgraph of  $G[A_2]$ ,  $G[B_3 - A_1] - S$  contains no  $H$ -immersion. Since  $G[A_1] - S$  contains no  $H$ -immersion,  $G[B_3 \cap A_1] - S$  does not contain  $H$ -immersion. Let  $Z_i$  be the set of edges between  $A_i$  and  $B_i$  for each  $i = 1, 2, 3$ . By Claim 1, there exists  $Z'$  with  $Z_1 \subseteq Z' \subseteq E(G)$  and  $|Z'| \leq |Z_1| + (2^h - 4)f'(kg(k))$  such that  $G[B_3] - (Z' \cup S)$  contains no  $H$ -immersion. Hence again by Claim 1, there exists  $Z''$  with  $Z' \cup Z_3 \subseteq Z'' \subseteq E(G)$  and  $|Z''| \leq |Z' \cup Z_3| + (2^h - 4)f'(kg(k)) \leq 2g(k) + 2(2^h - 4)f'(kg(k))$

such that  $G - (Z'' \cup S)$  contains no  $H$ -immersion. Let  $Z_S$  be the set of edges incident with vertices in  $S$ . Then  $G - (Z'' \cup Z_S)$  has no  $H$ -immersion, since  $H$  has no isolated vertices. We are done since  $Z'' \cup Z_S$  has size at most  $(k+1)g(k) + (2^{h+1} - 8)f'(kg(k)) \leq f(k, |S|)$ . So  $\mathcal{E}$  satisfies (E2).

Finally, if there exists  $[A, B] \in \mathcal{E}$  such that there are less than  $g(k)$  edges incident with  $B$ , then  $G[B] - (E(G[B]) \cup S)$  has no  $H$ -immersion. By Claim 1, there exists  $Z \subseteq E(G)$  with  $|Z| \leq 2g(k) + (2^h - 4)f'(kg(k))$  such that  $G - (Z \cup S)$  has no  $H$ -immersion. Since  $|Z \cup Z_S| \leq (k+1)g(k) + (2^h - 4)f'(kg(k))$ , it is a contradiction and hence  $\mathcal{E}$  satisfies (E3). Therefore,  $\mathcal{E}$  is an edge-tangle in  $G$  of order  $g(k)$ .  $\square$

Let  $T$  be the tree and  $\mathcal{P} = \{X_t : t \in V(T)\}$  the partition of  $V(G)$  satisfying Lemma 5.1. We call  $X_t$  the bag at  $t$ . For each edge  $e \in E(T)$ , there exists an edge-cut  $[A_e, B_e]$  of  $G$  such that each  $A_e$  and  $B_e$  is the union of the bags of the vertices in a components of  $T - e$ . So  $[A_e, B_e]$  has order at most three and the edges between  $A_e$  and  $B_e$  are the parallel edges with the same ends. Since  $\mathcal{E}$  is an edge-tangle of order greater than three,  $[A_e, B_e] \in \mathcal{E}$  or  $[B_e, A_e] \in \mathcal{E}$  but not both. If  $[A_e, B_e] \in \mathcal{E}$ , then we direct  $e$  such that  $B_e$  contains the bag of the head of  $e$ ; otherwise, we direct  $e$  in the opposite direction. Hence, we obtain an orientation of  $T$  and there exists a vertex  $t^*$  of  $T$  with out-degree zero.

**Claim 5:** There exist a set  $R$  of loops of  $G[X_{t^*}]$  with  $|R| \leq (k-1)hd$  and a set  $U \subseteq E(T)$  with  $|U| \leq (k-1)hd$  such that every edge in  $U$  is incident with  $t^*$ , and for every  $H$ -immersion  $(\pi_V, \pi_E)$  in  $G$ , one of the following holds.

- The image of  $\pi_E$  contains a non-loop edge of  $G[X_{t^*}]$ .
- The image of  $\pi_E$  contains an edge in  $R$ .
- The image of  $\pi_E$  is contained in  $G[A_e]$  or contains an edge of  $G$  between  $A_e$  and  $B_e$  for some  $e \in U$ .

**Proof of Claim 5:** By taking short-cuts of paths, for every  $H$ -immersion  $(\pi_V, \pi_E)$  in  $G$ , there exists an  $H$ -immersion  $(\pi'_V, \pi'_E)$  in  $G$  such that  $\pi'_V = \pi_V$  and the image of  $\pi'_E$  is contained in the image of  $\pi_E$  such that for every edge  $x$  of  $H$ , there are at most two edges  $e$  of  $T$  incident with  $t^*$  such that the image of  $\pi'_E(x)$  intersects  $G[A_e]$ . Therefore, for every  $H$ -immersion  $\pi = (\pi_V, \pi_E)$  in  $G$  in which the image of  $\pi_E$  does not intersect non-loop edges of  $G[X_{t^*}]$ , there exist an  $H$ -immersion  $(\pi'_V, \pi'_E)$  in  $G$  and a set  $W_\pi$  of edges of  $T$  incident with  $t^*$  and  $|W_\pi| \leq hd$  such that the image of  $\pi'_E$  does not intersect any non-loop

edges of  $G[\bigcap_{e \in W_\pi} B_e]$ . Hence, by an easy greedy algorithm, either there exist  $k$  edge-disjoint  $H$ -immersions  $\pi_1, \pi_2, \dots, \pi_k$  in  $G$  such that  $W_{\pi_1}, \dots, W_{\pi_k}$  are pairwise disjoint, or there exist at most  $(k-1)hd$  edges of  $T$  incident with  $t^*$  and at most  $(k-1)hd$  loops of  $G[X_{t^*}]$  such that for each  $H$ -immersion  $(\pi_V, \pi_E)$  in  $G$  where the image of  $\pi_E$  does not contain any non-loop edge of  $G[X_{t^*}]$ , either  $W_\pi$  intersects  $W_{\pi_i}$  for some  $1 \leq i \leq k$ , or the image of  $\pi_E$  contains one of the chosen loops of  $G[X_{t^*}]$ . The former is impossible since  $G$  does not contain  $k$  edge-disjoint  $H$ -immersions. Let  $U$  be the set of the edges of  $T$  and  $R$  be the loops of  $G$  mentioned in the latter case. Then  $U$  and  $R$  satisfy the conclusion of this claim.  $\square$

**Claim 6:** For every  $[A, B] \in \mathcal{E}$  of order less than  $g(k) - 3(k-1)hd$ ,  $G[B \cap X_{t^*}]$  contains at least  $f(k, |S|) - (k-1)g(k) - 4(k-1)hd$  non-loop edges.

**Proof of Claim 6:** Suppose that there exists  $[A, B] \in \mathcal{E}$  of order less than  $g(k) - 3(k-1)hd$  such that  $G[B \cap X_{t^*}]$  contains less than  $f(k, |S|) - (k-1)g(k) - 4(k-1)hd$  non-loop edges. Let  $U$  be the set of edges of  $T$  incident with  $t^*$  and  $R$  the set of loops of  $G[X_{t^*}]$  mentioned in Claim 5. Note that  $|U| \leq (k-1)hd$  and  $|R| \leq (k-1)hd$ . Let  $A' = A \cup \bigcup_{e \in U} A_e$  and  $B' = B \cap \bigcap_{e \in U} B_e$ . By (E1) and (E2),  $[A', B'] \in \mathcal{E}$ . By Claim 5, for every  $H$ -immersion  $(\pi_V, \pi_E)$  in  $G$ , one of the following holds.

- The image of  $\pi_E$  is contained in  $G[A']$ .
- The image of  $\pi_E$  contains a loop in  $R$  or an edge between  $A'$  and  $B'$ .
- The image of  $\pi_E$  contains a non-loop edge of  $G[X_{t^*}]$ .

By the definition of  $\mathcal{E}$ ,  $G[A'] - S$  has no  $H$ -immersion. Let  $Z_S$  be the set of edges incident with vertices  $S$ . So  $|Z_S| \leq (k-1)g(k)$ . Let  $Z_0$  be the set of the non-loop edges in  $G[X_{t^*}]$  and the edges of  $G$  between  $A'$  and  $B'$ . Let  $Z = Z_S \cup Z_0 \cup R$ . Therefore,  $G - Z$  has no  $H$ -immersion. But  $|Z| \leq (k-1)g(k) + x + |R| + 3(k-1)hd \leq f(k, |S|)$ , where  $x$  is the number of non-loop edges of  $G[X_{t^*}]$ , a contradiction.  $\square$

In particular, Claim 6 implies that  $X_{t^*}$  contains at least two vertices and  $G[X_{t^*}]$  is 4-edge-connected.

For every vertex  $v$  in  $X_{t^*}$ , define  $S_v$  to be the set of vertices  $u$  of  $G - X_{t^*}$  in which every path in  $G$  from  $u$  to  $X_{t^*}$  contains  $v$ . Note that  $S_v$  is empty if  $N_G(v) \subseteq X_{t^*}$ . Define  $\mathcal{E}'$  to be the set of edge-cuts  $[A', B']$  of  $G[X_{t^*}]$  of order less than  $g(k) - 3(k-1)hd$  such that  $[A', B'] \in \mathcal{E}'$  if and only if  $[A' \cup \bigcup_{v \in A'} S_v, B' \cup \bigcup_{v \in B'} S_v] \in \mathcal{E}$ . We claim that  $\mathcal{E}'$  is an edge-tangle of order

$g(k) - 3(k-1)hd$  in  $G[X_{t^*}]$ . Since the order of  $[A' \cup \bigcup_{v \in A'} S_v, B' \cup \bigcup_{v \in B'} S_v]$  equals the order of  $[A', B']$ ,  $\mathcal{E}'$  satisfies (E1). If there exist  $[A'_i, B'_i] \in \mathcal{E}'$  for  $i = 1, 2, 3$  such that  $A'_1 \cup A'_2 \cup A'_3 = X_{t^*}$ , then  $A'_1 \cup A'_2 \cup A'_3 \cup \bigcup_{v \in A'_1 \cup A'_2 \cup A'_3} S_v = V(G)$ , a contradiction. So  $\mathcal{E}'$  satisfies (E2). In addition,  $\mathcal{E}'$  satisfies (E3) since  $G[B \cap X_{t^*}]$  contains at least  $f(k, |S|) - (k-1)g(k) - 4(k-1)hd \geq (k+1)g(k) - 4(k-1)hd \geq g(k) - 3(k-1)hd$  non-loop edges for every  $[A, B] \in \mathcal{E}$  by Claim 6. Therefore,  $\mathcal{E}'$  is an edge-tangle of order  $g(k) - 3(k-1)hd$  in a 4-edge-connected graph  $G[X_{t^*}]$ .

Define  $\mathcal{E}_k$  and  $\mathcal{E}'_k$  to be the subsets of  $\mathcal{E}$  and  $\mathcal{E}'$  consisting of edge-cuts of order less than  $\theta_k$ , respectively. By Theorem 4.6,  $\mathcal{E}'_k$  controls a  $K_{w_k}$ -thorns  $\alpha$  in  $G[X_{t^*}]$ . Since  $\alpha$  is in  $G[X_{t^*}]$ ,  $\mathcal{E}_k$  controls  $\alpha$ . By Lemma 3.9, there exist  $Z^* \subseteq E(G)$  with  $|Z^*| \leq \xi_k$  and  $[A, B] \in \mathcal{E}_k \subseteq \mathcal{E}$  such that  $G[B] - Z^*$  has no  $H$ -immersion. On the other hand, every  $H$ -immersion in  $G$  intersects an edge incident with  $S$  by the definition of  $\mathcal{E}$ . Therefore, every  $H$ -immersion in  $G$  intersects an edge in  $Z^* \cup Z_S$ , where  $Z_S$  is the set of edges of  $G$  incident with  $S$ . By Claim 1, there exists  $Z^{**}$  with  $Z^{**} \subseteq E(G)$  and  $|Z^{**}| \leq |Z^*| + |Z_S| + g(k) + (2^h - 4)f'(kg(k)) \leq \xi_k + kg(k) + (2^h - 4)f'(kg(k)) \leq f(k, |S|)$  such that  $G - Z^{**}$  has no  $H$ -immersion, a contradiction. This completes the proof. ■

Theorem 1.1 is an immediate corollary of the following theorem, as every 4-edge-connected graph is nearly 4-edge-connected.

**Theorem 5.3.** *For every graph  $H$ , there exists a function  $f : \mathbb{N} \rightarrow \mathbb{N}$  such that for every nearly 4-edge-connected graph  $G$ , either  $G$  contains  $k$  edge-disjoint  $H$ -immersions, or there exists  $Z \subseteq E(G)$  with  $|Z| \leq f(k)$  such that  $G - Z$  contains no  $H$ -immersion.*

**Proof.** Let  $H'$  be the graph obtained from  $H$  by deleting all isolated vertices. For every positive integer  $k$ , define  $f(k)$  to be the number  $f_{5.2}(k, 0)$ , where  $f_{5.2}$  is the function mentioned in Theorem 5.2 by taking  $H$  to be  $H'$ .

Let  $G$  be a nearly 4-edge-connected graph. If  $|V(G)| < |V(H)|$ , then clearly  $G$  does not contain an  $H$ -immersion. So we may assume that  $|V(G)| \geq |V(H)|$ . Hence, for every  $H'$ -immersion  $(\pi'_V, \pi'_E)$  of  $G$ , we can extend  $\pi'_V$  to an injection  $\pi_V$  with domain  $V(H)$  by further mapping isolated vertices of  $H$  to some vertices of  $G$  not in the image of  $\pi'_V$  such that  $(\pi'_V, \pi_E)$  is an  $H$ -immersion in  $G$ . Therefore, for every integer  $k$ ,  $G$  contains  $k$  edge-disjoint  $H$ -immersions if and only if  $G$  contains  $k$  edge-disjoint  $H'$ -immersions.

Now fix  $k$  be a positive integer. If  $G$  does not contain  $k$  edge-disjoint  $H$ -immersions, then  $G$  does not contain  $k$  edge-disjoint  $H'$ -immersions, so there exists  $Z \subseteq E(G)$  with  $|Z| \leq f_{5.2}(k, 0) = f(k)$  such that  $G - Z$  has no  $H'$ -immersion by Theorem 5.2. But it implies that  $G - Z$  has no  $H$ -immersions. This proves the theorem. ■

Now we prove Theorem 1.2. The following is the restatement.

**Theorem 5.4.** *For every graph  $H$ , there exists a function  $f : \mathbb{N} \rightarrow \mathbb{N}$  such that for every positive integer  $k$ , every graph  $G$  either contains  $k$   $H$ -half-integral immersions  $(\pi_V^{(1)}, \pi_E^{(1)}), \dots, (\pi_V^{(k)}, \pi_E^{(k)})$  such that for each edge  $e$  of  $G$ , there exist at most two distinct pairs  $(i, e')$  with  $1 \leq i \leq k$  and  $e' \in E(H)$  such that  $e \in \pi_E^{(i)}(e')$ , or there exists  $Z \subseteq E(G)$  with  $|Z| \leq f(k)$  such that  $G - Z$  contains no  $H$ -half-integral immersion.*

**Proof.** For every graph  $R$ , define  $f_R$  to be the function  $f$  mentioned in Theorem 5.3 by taking  $H = R$ . Let  $c$  be the number of components of  $H$ . For every  $i$  with  $1 \leq i \leq c$ , define  $\mathcal{F}_i$  to be the set of graphs consisting of  $i$  components of  $H$ . For every positive integers  $m \geq 2$  and  $n$ , define  $f_1(n) = (n - 1) \max\{f_R(n) : R \in \mathcal{F}_1\}$  and define  $f_m(n) = (km - 1)f_H(k) + m^{km+m}m!f_{m-1}(n)$ . We claim that for every  $m$  with  $1 \leq m \leq c$ , for every graph  $W \in \mathcal{F}_m$ , and for every positive integer  $k$ , every graph  $G$  either contains  $k$   $W$ -half-integral immersions  $(\pi_V^{(1)}, \pi_E^{(1)}), \dots, (\pi_V^{(k)}, \pi_E^{(k)})$  such that for each edge  $e$  of  $G$ , there exist at most two distinct pairs  $(i, e')$  with  $1 \leq i \leq k$  and  $e' \in E(W)$  such that  $e \in \pi_E^{(i)}(e')$ , or there exists  $Z \subseteq E(G)$  with  $|Z| \leq f_m(k)$  such that  $G - Z$  has no  $W$ -half-integral immersion. We shall prove this claim by induction on  $m$ .

Let  $G$  be a graph, and let  $G'$  be the graph obtained from  $G$  by duplicating each edge. Note that every edge-cut of  $G'$  has even order. If  $[A, B]$  is an edge-cut of a component of  $G'$  of order less than four, then it has order two and the two edges between  $A$  and  $B$  are parallel edges with the same ends. So every component of  $G'$  is nearly 4-edge-connected.

Note that for every graph  $R$ ,  $G'$  contains  $k$  edge-disjoint  $R$ -immersions if and only if  $G$  contains  $k$   $R$ -half-integral immersions  $(\pi_V^{(1)}, \pi_E^{(1)}), \dots, (\pi_V^{(k)}, \pi_E^{(k)})$  such that for each edge  $e$  of  $G$ , there exist at most two distinct pairs  $(i, e')$  with  $1 \leq i \leq k$  and  $e' \in E(R)$  such that  $e \in \pi_E^{(i)}(e')$ . Similarly, if there exists  $Z' \subseteq E(G')$  such that  $G' - Z'$  has no  $R$ -immersion, then  $G - Z$  has no  $R$ -half-integral immersion, where  $Z$  is the set of edges of  $G$  which has a copy in  $Z'$ . Note that  $|Z| \leq |Z'|$ .



Let  $W$  be an arbitrary graph in  $\mathcal{F}_m$ , and let  $k$  be a positive integer. Suppose that  $G$  does not contain  $k$   $W$ -half-integral immersions  $(\pi_V^{(1)}, \pi_E^{(1)}), \dots, (\pi_V^{(k)}, \pi_E^{(k)})$  such that for each edge  $e$  of  $G$ , there exist at most two distinct pairs  $(i, e')$  with  $1 \leq i \leq k$  and  $e' \in E(W)$  such that  $e \in \pi_E^{(i)}(e')$ . So  $G'$  does not contain  $k$  edge-disjoint  $W$ -immersions. It suffices to show that there exists  $Z' \subseteq E(G')$  with  $|Z'| \leq f_m(k)$  such that  $G' - Z'$  has no  $W$ -immersion.

We first assume that  $m = 1$ . Let  $G'_1, G'_2, \dots, G'_p$  be the components of  $G'$  containing a  $W$ -immersion, and let  $k_i$  be the maximum number of edge-disjoint  $W$ -immersions in  $G'_i$  for  $1 \leq i \leq p$ . If  $\sum_{i=1}^p k_i \geq k$ , then  $G'$  contains  $k$  edge-disjoint  $W$ -immersions, a contradiction. So  $\sum_{i=1}^p k_i < k$ . In particular,  $p < k$ . By Theorem 5.3, for every  $1 \leq i \leq p$ , there exists  $Z'_i \subseteq E(G'_i)$  with  $|Z'_i| \leq f_H(k_i + 1)$  such that  $G'_i - Z'_i$  has no  $W$ -immersion. Therefore  $G' - Z'$  has no  $W$ -immersion, where  $Z' = \bigcup_{i=1}^p Z'_i \subseteq E(G')$ . Note that  $|Z'| \leq \sum_{i=1}^p f_H(k_i + 1) \leq (k - 1)f_H(k) = f_1(k)$ . This proves the base case of the induction.

Now we assume that our claim holds for every smaller  $m$ . Note that  $W$  has  $m$  components. Let  $W_1, W_2, \dots, W_m$  be the components of  $W$ . For every  $i$  with  $1 \leq i \leq m$ , define  $S_i$  to be the set of components of  $G'$  containing an  $W_i$ -immersion. If  $|S_i| \geq km$  for every  $i$  with  $1 \leq i \leq m$ , then  $G'$  contains  $km$  components  $G'_1, \dots, G'_{km}$  of  $G'$  such that  $G'_{(i-1)k+j} \in S_i$  for each  $i$  with  $1 \leq i \leq m$  and each  $j$  with  $1 \leq j \leq k$ . So  $G'$  contains  $k$  edge-disjoint  $W$ -immersions, a contradiction. Therefore, there exists  $t$  with  $1 \leq t \leq m$  such that  $|S_t| < km$ .

Define  $L$  to be the disjoint union of the components of  $G'$  in  $S_t$ , and define  $R = G' - V(L)$ . Note that  $R$  has no  $W_t$ -immersion by the definition of  $S_t$ . If  $L$  does not contain  $k$  edge-disjoint  $W_t$ -immersions, then there exists  $Z_t \subseteq E(L)$  with  $|Z_t| \leq g_1(k)$  such that  $L - Z_t$  has no  $W_t$ -immersion. Since  $W_t$  is connected,  $G'$  has no  $W_t$ -immersion and hence has no  $W$ -immersion. So we are done in this case.

Now we assume that  $L$  contains  $k$  edge-disjoint  $W_t$ -immersions. Note that  $R$  does not contain  $k$  edge-disjoint  $(W - V(W_t))$ -immersions, otherwise  $G'$  contains  $k$  edge-disjoint  $W$ -immersions. Note that  $W - V(W_t) \in \mathcal{F}_{m-1}$ . By the induction hypothesis, there exists  $Z_R \subseteq E(G')$  with  $|Z_R| \leq f_{m-1}(k)$  such that  $R - Z_R$  has no  $(W - V(W_t))$ -immersion. So  $R - Z_R$  has no  $W$ -immersion. On the other hand, for each component  $C$  of  $L$ ,  $C$  is nearly 4-edge-connected and has no  $k$  edge-disjoint  $W$ -immersions, so there exists  $Z_C \subseteq E(C)$  with  $|Z_C| \leq f_H(k)$  such that  $C - Z_C$  has no  $W$ -immersion.

Define  $Z_0 = Z_R \cup \bigcup Z_C$ , where the second union is taken over all components  $C$  of  $L$ . Therefore,  $|Z_0| \leq f_{m-1}(k) + (km - 1)f_H(k)$ , and  $R - Z_0$  and  $C - Z_0$  do not contain a  $W$ -immersion for every component  $C$  of  $L$ .

Let  $\ell$  be the number of components of  $L$ . Define  $Q_0 = R$  and define  $Q_i$  to be the  $i$ -th component of  $L$ , for every  $1 \leq i \leq \ell$ . Note that  $Q_i - Z_0$  has no  $W$ -immersion for every  $0 \leq i \leq \ell$ . We say that  $(P_0, P_1, \dots, P_\ell)$  is a  $(\ell+1)$ -partition of  $[m]$  if  $P_0, P_1, \dots, P_\ell$  are pairwise disjoint (possibly empty) proper subsets of  $[m]$  with  $\bigcup_{i=0}^\ell P_i = [m]$ . Since  $G'$  has no  $k$  edge-disjoint  $W$ -immersions, for every  $(\ell+1)$ -partition  $\mathcal{P} = (P_0, \dots, P_\ell)$  of  $[m]$ , there exists  $j$  with  $0 \leq j \leq \ell$  such that  $Q_j$  does not contain  $k$  edge-disjoint  $(\bigcup_{i \in P_j} W_i)$ -immersions, so there exists  $Z_{\mathcal{P}} \subseteq E(Q_j)$  with  $|Z_{\mathcal{P}}| \leq f_{|P_j|}(k)$  such that  $Q_j - Z_{\mathcal{P}}$  has no  $(\bigcup_{i \in P_j} W_i)$ -immersions. Define  $Z^*$  to be the union of  $Z_0$  and  $Z_{\mathcal{P}}$  for all  $(\ell+1)$ -partitions  $\mathcal{P}$  of  $[m]$ . Note that  $|Z^*| \leq |Z_0| + (m^{m+\ell+1}m! - 1) \max\{f_j(k) : 1 \leq j \leq m-1\} \leq f_{m-1}(k) + (km - 1)f_H(k) + (m^{km+m}m! - 1)f_{m-1}(k) \leq f_m(k)$ .

We claim that  $G' - Z^*$  has no  $W$ -immersion. Suppose that  $G' - Z^*$  contains a  $W$ -immersion. Since  $Q_i - Z^*$  has no  $W$ -immersion for every  $0 \leq i \leq \ell$ , there exists a  $(\ell+1)$ -partition  $\mathcal{P} = (P_0, P_1, \dots, P_\ell)$  of  $[m]$  such that  $Q_j - Z^*$  contains a  $(\bigcup_{i \in P_j} W_i)$ -immersion for every  $0 \leq j \leq \ell$ . However, it contradicts the definition of  $Z_{\mathcal{P}}$ . This completes the proof of the claim. Consequently,  $f_c$  is the function satisfies the conclusion of this theorem. ■

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